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A Note on Dagum Sigmoid Function with Applications to Income and Lifetime Data

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Abstract: The Dagum distribution is a flexible and simple model with applications to income, financial and lifetime data.

We prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function by a class of Dagum cumulative distribution function – (DCDF). Numerical examples, illustrating our results are given.

Keywords: Dagum cumulative distribution function (DCDF), shifted Heaviside function, Hausdorff distance, upper and lower bounds

I. INTRODUCTION

Dagum (1977) [2] motivates his model from the empirical observation that the income elasticity $\eta(F,t)$ of the cumulative distribution function (CDF) F of income is a decreasing and bounded function F.

Starting from the differential equation

$$\eta(F,t) = \frac{d\log F(t)}{d\log t} = ap\left(1-\left(F(t)\right)^{\frac{1}{p}}\right),\ t \geq 0, \tag{1}$$

subject to $p > 0$ and $ap > 0$, one obtains (see, also Kleiber [1]):

$$F(t) = \left(1+\left(\frac{t}{b}\right)^{-a}\right)^{-p}. \tag{2}$$

This approach was further developed in a series of papers on generating systems for income distribution [3]–[6].

For other results, see [7], [8], [9] and [10].

In this paper we prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function by a class of Dagum cumulative distribution function – (DCDF).

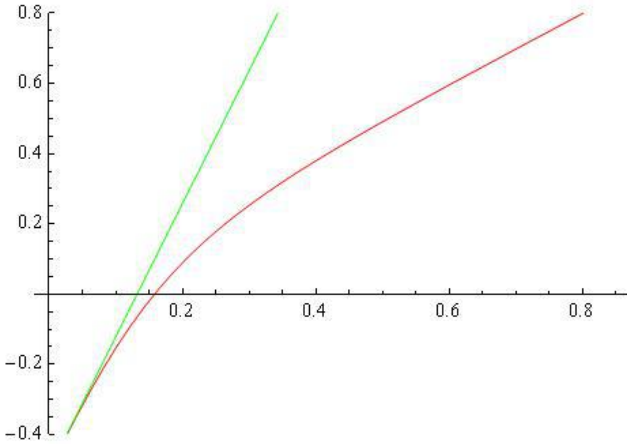

Figure 1: The functions H and G.

II. PRELIMINARIES

Definition 1. *The (basic) step function is:*

$$\tilde{h}_{t_0}(t) = \begin{cases} 0, & \text{if } \quad t < t_0, \\ [0,1], & \text{if } \quad t = t_0, \\ 1, & \text{if } \quad t > t_0, \end{cases} \qquad (3)$$

Definition 2. *[11], [12] The Hausdorff distance (the H–distance) [11] $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathrm{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathrm{R}$.*

More precisely,

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} \| A - B \|, \sup_{B \in F(g)} \inf_{A \in F(f)} \| A - B \| \}, \qquad (4)$$

wherein $\|.\|$ is any norm in R^2, e. g. the maximum norm $\|(t,x)\| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in R^2 is $\| A - B \| = max(| t_A - t_B |, | x_A - x_B |)$.

Let us point out that the Hausdorff distance is a natural measuring criteria for the approximation of bounded discontinuous functions.

III. MAIN RESULTS

Let us consider the following three parametric sigmoid function

$$F^*(t) = \left(1 + \left(\frac{t}{b} \right)^{-a} \right)^{-p} \qquad (5)$$

with

$$F^*(t_0) = \frac{1}{2}, \; t_0 = b \left(\left(\frac{1}{2} \right)^{-\frac{1}{p}} - 1 \right)^{-\frac{1}{a}}. \qquad (6)$$

The H-distance $d = \rho(h_{t_0}, F^*)$ between the shifted Heaviside step function h_{t_0} and the sigmoidal function F^* satisfies the relation:

$$F^*(t_0 + d) = \left(1 + \left(\frac{t_0 + d}{b} \right)^{-a} \right)^{-p} = 1 - d. \qquad (7)$$

The following theorem gives upper and lower bounds for $d = d(a,b,p)$

Theorem 1. *Let*

$$\alpha = -\frac{1}{2}; \; \beta = 1 + \frac{ap}{b} \left(\frac{1}{2} \right)^{\frac{1+p}{p}} \left(\left(\frac{1}{2} \right)^{-\frac{1}{p}} - 1 \right)^{\frac{1+a}{a}}. \qquad (8)$$

The H-distance d between the function h_{t_0} and the function F^* can be expressed in terms of the *parameters* a, b, p *for any real* $\beta \geq \frac{e^{1.05}}{2.1} \approx 1.36079$ *as follows:*

$$d_l = \frac{1}{2.1\beta} < d < \frac{\ln(2.1\beta)}{2.1\beta} = d_r. \tag{9}$$

Proof. We define the functions

$$H(d) = F^*(t_0 + d) - 1 + d = \left(1 + \left(\frac{t_0 + d}{b}\right)^{-a}\right)^{-p} - 1 + d \tag{10}$$

$$G(d) = \alpha + \beta d. \tag{11}$$

From Taylor expansion

$$H(d) - G(d) = O(d^2)$$

we see that the function $G(d)$ approximates $H(d)$ with $d \to 0$ as $O(d^2)$ (cf. Fig. 1).

In addition $G'(d) > 0$ and for $\beta \geq 1.36079$

$$G(d_l) < 0; \; G(d_r) > 0.$$

This completes the proof of the inequalities (9).

The generated sigmoidal functions $F^*(t)$ for $a = 10.5; b = 0.9; p = 0.8$; $a = 16; b = 0.7; p = 1.1$ and $a = 25; b = 0.6; p = 1.2$ are visualized on Fig. 2–Fig. 4.

From the Fig. 2–Fig.4 it can be seen that the "supersaturation" is fast.

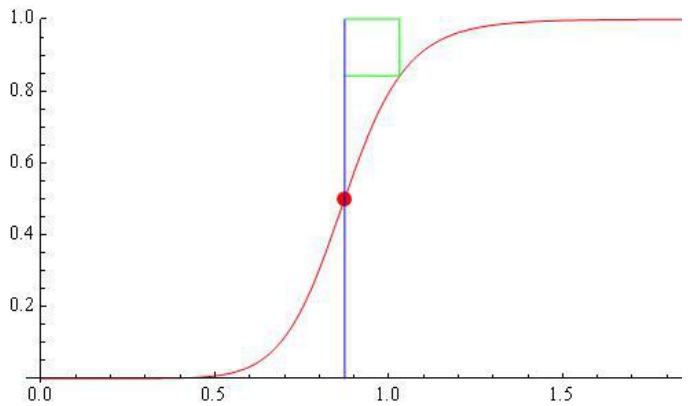

Figure 2: The function $F^*(t)$ for $a = 10.5; b = 0.9; p = 0.8$ $t_0 = 0.872908$: H-distance $d = 0.158109$; $d_l = 0.125693$; $d_r = 0.260676$.

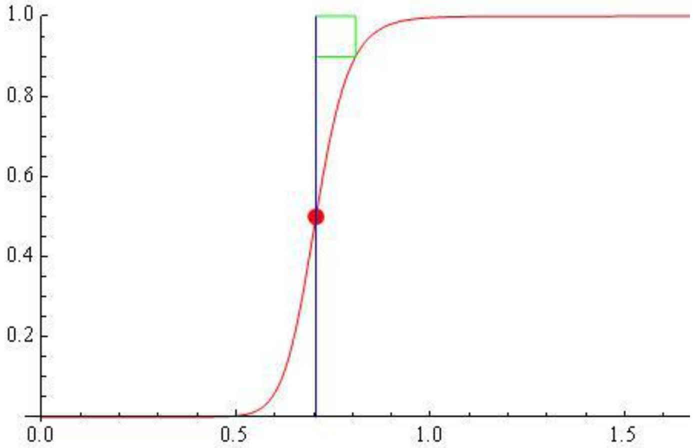

Figure 3: The function $F^*(t)$ for $a=16; b=0.7; p=1.1$ $t_0=0.705722$: H-distance $d=0.101521$; $d_l=0.0697283$; $d_r=0.185697$.

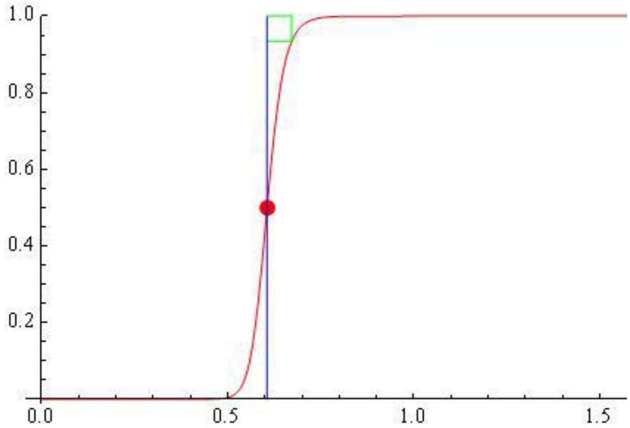

Figure 4: The function $F^*(t)$ for $a=25; b=0.6; p=1.2$ $t_0=0.605937$: H-distance $d=0.0661316$; $d_l=0.0401451$; $d_r=0.129077$.

IV. CONCLUSION

In this paper we prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function by a class of Dagum cumulative distribution function – (DCDF).

A family of parametric sigmoidal functions based on Dagum cumulative distribution function – (DCDF) is introduced finding application in income, financial and lifetime theory.

In this note we consider dependence of supersaturation by means of this class.

We prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function h_{t_0} and the function F^*.

Numerical examples, illustrating our results are given.

We propose a software module (intellectual property) within the programming environment *CAS Mathematica* for the analysis of the considered family of (DCDF) functions.

For other results, see [13]–[25].

V. ACKNOWLEDGEMENTS

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