International Journal of Latest Research in Engineering and Technology (IJLRET)
ISSN: 2454-5031
www.ijlret.com || Volume 05 - Issue 02 || February 2019 || PP. 01-16

DiffusionAdvectionReactionEquation in ConditionsBorder Open Solutionby Fourier-Laplace Transform and itsComparisonWith an Application Dispersion of Air Pollutants

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Abstract: The advection diffusion reaction equation is an important expression within many branches of engineering, for its wide use in mathematical modeling processes dispersion and diffusion mass transport, economy or contaminant within a porous medium or atmospheric. The aim of this article is to show the relationship they have two solutions to solve the partial differential equation presented without borders and may be an option analytical solution for one-dimensional models using Fourier transform - Laplace have Fourier integral with forcing, which contains a Delta function. Examples of cups Pollutant modeled in time for this case.
Keywords: Fourier transform, Dirac delta function, advection diffusion reaction equation, Laplace transform and Fourier integral.

1. Introduction: Pollutant Dispersion Model

A pollutant contained in ambient air is subject to various physical and chemical processes that influence propagation and. Some of these processes are: transport advection, sedimentation, turbulent diffusion and transformation by various chemical reactions, also due to the complexity of these processes, the dispersion of each substance by fixed or mobile sources has been emitted into the atmosphere is a dimensional and nonstationary phenomenon. In what follows a linear model that takes into account these processes occurs, and assumes that all of the coefficients contained in the respective settings are known. The spread of pollutants in the atmosphere can be described by the following equation (Parra-Guevara, D. & Skiba, Y. N, 2003 [5]),$\boldsymbol{\varphi}\mu\boldsymbol{\sigma}\mu_z\boldsymbol{f}(\boldsymbol{r},\boldsymbol{t})$

$$\frac{\partial\varphi}{\partial t}+U.\nabla\varphi-\nabla.(\mu\nabla\varphi)-\frac{\partial}{\partial z}\left(\mu_z\frac{\partial\varphi}{\partial z}\right)+\sigma\varphi=f(r,t) \qquad \textbf{(1.1)}$$

With Without Borders conditions as

$$\varphi(r,0)=\varphi^0(r)$$

With

$$\varphi(-\infty<x<+\infty,t)=0t\geq 0$$
$$\varphi(-\infty<y<+\infty,t)=0t\geq 0$$
$$\varphi(0<z<+\infty,t)=0t\geq 0$$

So we have the three components

$$\frac{\partial\varphi}{\partial t}+u\frac{\partial\varphi}{\partial x}-D\frac{\partial^2\varphi}{\partial x^2}+\sigma\varphi=\gamma \qquad \textbf{(1.2)}$$

$$\varphi(-\infty<x<+\infty,t)=0t\geq 0$$

$$\varphi(x,t0)=\emptyset(x)$$

$$\frac{\partial\varphi}{\partial t}+v\frac{\partial\varphi}{\partial y}-D\frac{\partial^2\varphi}{\partial y^2}+\sigma\varphi=\gamma \qquad \textbf{(1.3)}$$

$$\varphi(-\infty < y < +\infty, t) = 0 t \geq 0$$

$$\varphi(y, t0) = \emptyset(y)$$

$$\frac{\partial \varphi}{\partial t} + w\frac{\partial \varphi}{\partial z} - D\frac{\partial^2 \varphi}{\partial z^2} + \sigma\varphi = \gamma \qquad (1.4)$$

$$\varphi(0 < z < +\infty, t) = 0 t \geq 0$$

$$\varphi(z, t0) = \emptyset(z)$$

Solving Partial Differential Equations using these techniques is very important and given its versatility in solving these models can make use of methodologies to find a solution, we'll start with some properties of Fourier Transform.

Jordan's motto

The Fourier transform integral is defined as infinite according to ([2], HF Weinberger, Ed. Blaisdell Publishing Company, 1965])

$$u(x) = \int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx$$

Investment theorem: If f (x) is integrable in a square, then the Fourier transform of the Fourier transform is $2\pi f(-x)$

$$u(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega$$

Thus we have the following ways to solve EDP

$$F\left[\frac{\partial u}{\partial t}\right] = \frac{\partial u^*}{\partial t}$$

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = -\omega^2 u^*$$

Now for the Laplace transform according to [1]

$$L[f](s) = F[f](is)$$

$$L[f] = \int_{0}^{+\infty} f(x)e^{-sx}dx$$

With its inverse transform as

$$L[f] = \frac{1}{2i\pi}\int_{s-iL}^{s+iL} F(s)e^{sx}ds$$

This formula is known as inversion theorem Mellin. With the following applications.

$$L\left[\frac{\partial u}{\partial t}\right] = sU - u(x, 0)$$

$$L\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{\partial^2 U}{\partial x^2}$$

According to these applications have the diffusion equation advection reaction in its dimensional shape, the convection-diffusion equation is a partial differential equation derived from the parabolic type, which describes the physical phenomenon where particles or energy (or other physical quantities) is transformed into a physical system because of two processes: diffusion and convection. In its simplest form (where the diffusion coefficient and convection velocity are constant and there are no sources or sinks).

The two spatial terms on the left side (1.1) represent different physical processes: the second corresponds to the normal broadcast while the first variation describes the advection - so the equation is also known as advection-diffusion equation. Furthermore is the variable of interest, the constant D is the diffusion coefficient, and v is the flow rate.$\boldsymbol{\varphi}$

Advection diffusion reaction equation which abbreviate as EDAR

$$\frac{\partial \varphi}{\partial t} + v\frac{\partial \varphi}{\partial x} - D\frac{\partial^2 \varphi}{\partial x^2} + \sigma\varphi = \gamma \qquad \textbf{(1.1)}$$

$$\varphi(-\infty < x < +\infty, t) = 0 t \geq 0$$

$$\varphi(x, t0) = \emptyset(x)$$

To solve the equation we have to reduce to a simpler way so apply a known transformation function (J.R Jiménez Zenteno [1])

$$\varphi(x,t) = w(x,t)e^{rx-st} + e^{rx}H(x,t) \qquad \textbf{(1.5)}$$

Where H is the inhomogeneous part. By applying the transformation we get the following, the two coefficients of the exponential function dependent on wind speed and turbulence of the medium:

$$r = \frac{v}{2D} ys = (\frac{v^2}{4D} + \sigma)$$

And now the reduced equation is:

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} + e^{st}F(x,t) con F(x,t) = \gamma e^{-rx} - \frac{\partial H}{\partial t} + D\frac{\partial^2 H}{\partial x^2} \qquad \textbf{(1.6)}$$

Now with H = 0 we have the following, the term H would contain the nonhomogeneous conditions of the reduced equation but for this case it is not necessary.

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} + [\gamma e^{-rx+st}]$$

$$P(x,t) = [\gamma e^{-rx+st}]$$

2. Solution by Fourier Transform

Now we will proceed to solve by the Fourier Transform to know the solution of the reduced equation and its behavior with the data of the examples that will be shown later.

$$F\left[\frac{\partial w}{\partial t}\right] = DF\left[\frac{\partial^2 w}{\partial x^2}\right] + F[P(x,t)] \qquad \textbf{(2.0)}$$

So the equation is like

$$\frac{\partial u(\omega,t)}{\partial t} + D\omega^2(\omega,t) = u(\omega,t)^* \qquad \textbf{(2.1)}$$

$$u(\omega,t) = 0$$

Resolving the Non-homogeneous Linear D.O.E gives us the solution as

$$\boldsymbol{u(\omega,t) = e^{-D\omega^2 t}\int_{-\infty}^{+\infty} u(\omega,t)^* e^{D\omega^2 t} dt} \tag{2.2}$$

Now taking the Fourier Inverse Transform we have to

$$\boldsymbol{F^{-1}[u(\omega,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{-D\omega^2 t}\int_{-\infty}^{+\infty} u(\omega,t)^* e^{D\omega^2 t} dt\right] e^{-i\omega x} d\omega} \tag{2.3}$$

The Solution of w is now

$$\boldsymbol{w(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\left[\int_0^t e^{-D\omega^2(t-\tau)}\int_{-\infty}^{+\infty} P(\varepsilon,\tau)e^{i\omega\varepsilon} d\varepsilon\right] d\tau d\omega} \tag{2.4}$$

Now we remember that we have the function of reduction but without the term Non-homogeneous

$$\boldsymbol{\varphi(x,t) = w(x,t)e^{rx-st}}$$

$$\boldsymbol{\varphi(x,t) = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\left[\int_0^t e^{-D\omega^2(t-\tau)}\int_{-\infty}^{+\infty} P(\varepsilon,\tau)e^{i\omega\varepsilon} d\varepsilon\right] d\tau d\omega\right) e^{rx-st}} \tag{2.5}$$

Now we have this forcing within the previous solution which we must resolve

$$\boldsymbol{P(x,t) = \gamma e^{-rx+st}} \qquad \textbf{Con } \boldsymbol{\gamma = \mathrm{Q(t)}\delta(x - x0)}$$

(2.6)

Then we have the first comprehensive and introducing the exponential term of the transformation we have

$$\boldsymbol{\int_{-\infty}^{+\infty} \mathrm{Q}(\tau)\delta(\varepsilon - \varepsilon 0)e^{i\omega\varepsilon} d\varepsilon} \tag{2.7}$$

$$\boldsymbol{\int_{-\infty}^{+\infty} \mathrm{Q}(\tau)\delta(\varepsilon - \varepsilon 0)e^{i\omega\varepsilon + s\tau - r\varepsilon} d\varepsilon} \tag{2.8}$$

Now see this is the transform of the function P

$$\boldsymbol{\mathrm{Q}(\tau)e^{s\tau}\int_{-\infty}^{+\infty} e^{i\omega\varepsilon - r\varepsilon}\delta(\varepsilon - \varepsilon 0) d\varepsilon \;=\; \mathrm{Q}(\tau)e^{s\tau}\left[e^{-(r-i\omega)\varepsilon 0}\right]} \tag{2.9}$$

Now taking the inverse Fourier transform to obtain the following:

$$\boldsymbol{\int_{-\infty}^{+\infty} e^{-(r-i\omega)\varepsilon 0 - i\omega\varepsilon} d\omega \;=\; e^{-r\varepsilon 0}\int_{-\infty}^{+\infty} e^{-i\omega(\varepsilon-\varepsilon 0)} d\omega = \delta(\varepsilon - \varepsilon 0)e^{-r\varepsilon 0}} \tag{2.10}$$

$$\boldsymbol{= \delta(x - x0)e^{-rx0}}$$

This whole right side is by definition the Dirac delta function.

Now we still have the integral ty cup so we can do the following

$$\boldsymbol{\int_0^t Q(\tau)e^{s\tau} d\tau = \frac{1}{s}[e^{st} - 1]Q(t)} \tag{2.11}$$

Now finally the last integral we have

$$\int_{-\infty}^{+\infty} e^{-i\omega x-(t-\tau)D\omega^2} d\omega d\tau \tag{2.12}$$

With the following

$$s = (t - \tau) \tag{2.13}$$

$$a = ix$$

$$\int_{-\infty}^{+\infty} e^{-a\omega - sD\omega^2} d\omega d\tau$$

This is a type of Gaussian Integral which the solution is:

$$I(x,t) = \frac{1}{2\sqrt{\pi D(t-\tau)}} e^{-\frac{x^2}{4D(t-\tau)}} \tag{2.14}$$

Now we still Integrate Tao therefore we have to

$$\frac{1}{2\sqrt{\pi D}} \int_0^t \frac{e^{-\frac{x^2}{4D(t-\tau)}}}{(t-\tau)^2} d\tau \tag{2.15}$$

Then is the following and drifting about Tao

$$u = -\frac{x^2}{4D(t-\tau)} \; ; \; d\tau = -\frac{4D(t-\tau)^2}{x^2} du \tag{2.16}$$

$$= -\frac{4D}{x^2} \int_0^t e^u du = \frac{4De^{-\frac{x^2}{4Dt}}}{x^2}$$

So now the general solution is

$$\varphi(x,t) = \left(\frac{1}{2\sqrt{\pi D}} \frac{4De^{-\frac{x^2}{4Dt}}}{x^2} \frac{1}{s} [e^{st} - 1] Q(t) \delta(x - x0) e^{-rx0} \right) e^{rx - st} \tag{2.17}$$

To remove the Delta function simply integrate with respect to x, is defined as x0 and open border is within the range we can do so giving.

$$\varphi(x,t) = \left(\frac{1}{2\sqrt{\pi D}} \frac{4De^{-\frac{x^2}{4Dt}}}{x^2} \frac{1}{s} [1 - e^{-st}] Q(t) \delta(x - x0) e^{-r(x-x0)} \right) \tag{2.18}$$

$$\varphi(x,t) = \left(\frac{1}{2\sqrt{\pi D}} \int_{-\infty}^{\infty} \frac{4De^{-\frac{x^2}{4Dt}}}{x^2} \delta(x - x0) dx \right) \frac{1}{s} [1 - e^{-st}] Q(t) e^{-r(x-x0)}$$

And so

$$\varphi(x,t)=\left(\frac{1}{2\sqrt{\pi D}}\frac{4De^{-\frac{x0^2}{4Dt}}}{x0^2}\frac{1}{s}[1-e^{-st}]Q(t)e^{-r(x-x0)}\right) \quad (2.19)$$

And preserves the initial condition at t = 0. With

$$\varphi(x,t)=\left(2\sqrt{\frac{D}{\pi x0}}e^{-\frac{(x0)^2}{4Dt}-(\frac{v}{2D})x+(\frac{v}{2D})x0}\frac{1}{(\frac{v^2}{4D}+\sigma)}\left[1-e^{-(\frac{v^2}{4D}+\sigma)t}\right]Q(t)\right) \quad (2.20)$$

3. Laplace Transform Solution

Now we proceed to solve the Laplace Transform

$$L\left[\frac{\partial w}{\partial t}\right]=DL\left[\frac{\partial^2 w}{\partial x^2}\right]+L[P(x,t)] \quad (3.0)$$

The transformed equation is:

$$\frac{d^2w}{dx^2}-\frac{s}{D}w=0$$

$$w(0,s)=\emptyset(x)$$

So the solution in its homogeneous form is

$$\varphi(x,t)=\int_{-\infty}^{\infty}\frac{\emptyset(\varepsilon)e^{\frac{-(x-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}}d\varepsilon \quad (3.1)$$

Now at the beginning of Duhamel (**[8] Özisik)**which it is the sum of the Homogeneous and Particular solutions

$$S(x,t)=\int_{-\infty}^{\infty}\frac{e^{\frac{-(x-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}}d\varepsilon \quad (3.2)$$

Integrates in x, t and both solutions have the same core, so we have

$$w(x,t)=\int_{-\infty}^{\infty}S(x,t)\emptyset(\varepsilon)d\varepsilon+\int_0^t\int_{-\infty}^{\infty}S(x-\varepsilon,t-\tau)P(\varepsilon,\tau)d\varepsilon d\tau \quad (3.3)$$

Inhomogeneous part is

$$\varphi(x,t)=\int_{\varepsilon=-\infty}^{\infty}\int_0^t\frac{P(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}}e^{\frac{-(x-\varepsilon)^2}{4D(t-\tau)}}d\tau d\varepsilon \quad (3.4)$$

And so the general solution (Kevorkian [3]):

$$\varphi(x,t)=\int_{-\infty}^{\infty}\frac{\emptyset(\varepsilon)e^{\frac{-(x-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}}d\varepsilon+\int_{\varepsilon=-\infty}^{\infty}\int_0^t\frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}}e^{\frac{-(x-\varepsilon)^2}{4D(t-\tau)}}d\tau d\varepsilon \quad (3.5)$$

Recalling that the General has Transformation

$$\varphi(x,t) = w(x,t)e^{rx-st}$$

$$\varphi(x,t) = \left[\int_{-\infty}^{\infty} \frac{\emptyset(\varepsilon)e^{\frac{-(x-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}} d\varepsilon + \int_{\varepsilon=-\infty}^{\infty}\int_0^t \frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}} e^{\frac{-(x-\varepsilon)^2}{4D(t-\tau)}} d\tau d\varepsilon\right] e^{rx-st} \quad (3.6)$$

Recalling now

$$P(x,t) = Q(t)\delta(x-x0)e^{-rx+st}$$

And the initial condition zero

$$\int_{\varepsilon=-\infty}^{\infty}\int_0^t \frac{(Q(\tau)\delta(\varepsilon-\varepsilon 0)e^{-r\varepsilon+s\tau})}{\sqrt{4\pi D(t-\tau)}} e^{\frac{-(x-\varepsilon)^2}{4D(t-\tau)}} d\tau d\varepsilon \quad (3.7)$$

$$\int_0^t Q(\tau)e^{s\tau}\left(\int_{\varepsilon=-\infty}^{\infty} \frac{(\delta(\varepsilon-\varepsilon 0)e^{-r\varepsilon})}{\sqrt{4\pi D(t-\tau)}} e^{\frac{-(x-\varepsilon)^2}{4D(t-\tau)}} d\varepsilon\right) d\tau \quad (3.8)$$

By properties of Delta must be

$$\int_0^t Q(\tau)e^{s\tau}\left(\frac{e^{\frac{-(x-\varepsilon 0)^2}{4D(t-\tau)}-r\varepsilon 0}}{\sqrt{4\pi D(t-\tau)}}\right) d\tau = \frac{e^{-r\varepsilon 0}}{2\sqrt{\pi D}}\int_0^t Q(\tau)e^{s\tau}\frac{e^{\frac{-(x-\varepsilon 0)^2}{4D(t-\tau)}}}{(t-\tau)^2} d\tau \quad (3.9)$$

This can be put as follows

$$\int_0^t Q(\tau)e^{s\tau}\int_0^t \frac{e^{\frac{-(x-\varepsilon 0)^2}{4D(t-\tau)}}}{(t-\tau)^2} d\tau \quad (3.10)$$

Having the same portion found above the cup
Contaminant

$$\int_0^t Q(\tau)e^{s\tau} d\tau = \frac{1}{s}[e^{st}-1]Q(t)$$

And the second integral is similar to the previous we had obtained with x0 =ε0

$$\frac{1}{2\sqrt{\pi D}}\int_0^t \frac{e^{-\frac{(x-\varepsilon 0)^2}{4D(t-\tau)}}}{(t-\tau)^2} d\tau = \quad (3.11)$$

So the general solution is

$$\varphi(x,t) = \left(\frac{1}{2\sqrt{\pi D}}\frac{4De^{-\frac{(x-x0)^2}{4Dt}}}{(x-x0)^2}\frac{1}{s}[e^{st}-1]Q(t)\right)e^{rx-st} \quad (3.12)$$

$$\varphi(x,t)=\left(2\sqrt{\frac{D}{\pi}}\left(\frac{e^{-\frac{(x-x0)^2}{4Dt}+rx}}{(x-x0)^2}\right)\frac{1}{s}[1-e^{-st}]Q(t)\right)$$

$$\varphi(x,t)=\left(2\sqrt{\frac{D}{\pi}}\left(\frac{e^{-\frac{(x-x0)^2}{4Dt}+(\frac{v}{2D})x}}{(x-x0)^2}\right)\frac{1}{(\frac{v^2}{4D}+\sigma)}\left[1-e^{-(\frac{v^2}{4D}+\sigma)t}\right]Q(t)\right) \tag{3.13}$$

We now have two solutions

Fourier Transform

$$\varphi(x,t)=\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega x}\left[\int_0^t e^{-D\omega^2(t-\tau)}\int_{-\infty}^{+\infty}P(\varepsilon,\tau)e^{i\omega\varepsilon}d\varepsilon\right]d\tau d\omega\right)e^{rx-st}$$

$$\varphi(x,t)=\left(2\sqrt{\frac{D}{\pi x0}}e^{-\frac{(x0)^2}{4Dt}-(\frac{v}{2D})x+(\frac{v}{2D})x0}\frac{1}{(\frac{v^2}{4D}+\sigma)}\left[1-e^{-(\frac{v^2}{4D}+\sigma)t}\right]Q(t)\right)$$

Laplace Transform

$$\varphi(x,t)=\left[\int_{-\infty}^{\infty}\frac{\emptyset(\varepsilon)e^{\frac{-(x-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}}d\varepsilon+\int_{\varepsilon=-\infty}^{\infty}\int_0^t\frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}}e^{\frac{-(x-\varepsilon)^2}{4D(t-\tau)}}d\tau d\varepsilon\right]e^{rx-st}$$

$$\varphi(x,t)=\left(2\sqrt{\frac{D}{\pi}}\left(\frac{e^{-\frac{(x-x0)^2}{4Dt}+(\frac{v}{2D})x}}{(x-x0)^2}\right)\frac{1}{(\frac{v^2}{4D}+\sigma)}\left[1-e^{-(\frac{v^2}{4D}+\sigma)t}\right]Q(t)\right)$$

$$\varphi(x,t)=\left(2\sqrt{\frac{D}{\pi(x-x0)}}e^{-\frac{(x-x0)^2}{4Dt}+(\frac{v}{2D})x}\frac{1}{(\frac{v^2}{4D}+\sigma)}\left[1-e^{-(\frac{v^2}{4D}+\sigma)t}\right]Q(t)\right)$$

Now solving Fourier and Laplace are very similar except integration delta, which only the Gaussian integral gives the result

$$IF(x,t)=\left(2\sqrt{\frac{D}{\pi x0}}e^{-\frac{(x0)^2}{4Dt}-(\frac{v}{2D})x+(\frac{v}{2D})x0}\right) \tag{3.14}$$

If we substitute and the IF, or have the following$x=\varepsilon+x0x0=x-\varepsilon$

$$IF(x,t)=\left(2\sqrt{\frac{D}{\pi x0}}e^{-\frac{(x0)^2}{4Dt}-(\frac{v}{2D})(\varepsilon+x0)+(\frac{v}{2D})x0}\right)=\left(2\sqrt{\frac{D}{\pi x0}}e^{-\frac{(x0)^2}{4Dt}+(\frac{v}{2D})\varepsilon}\right) \tag{3.15}$$

$$= \left(2\sqrt{\frac{D}{\pi(x-\varepsilon)}}\, e^{-\frac{(x-\varepsilon)^2}{4Dt}+\left(\frac{v}{2D}\right)\varepsilon} \right) \quad (3.16)$$

We left at exactly Laplace to make that change and epsilon would have two representations and solution.

4. General solution of Dispersion Model Pollutant found:

With the General Transformation it has in its three directions found by solving Laplace, having the homogeneous medium in three directions has the following:

$$\varphi(x,t) = w(x,t)e^{rx-st}$$

$$\varphi(x,t) = \left[\int_{-\infty}^{\infty} \frac{\emptyset(\varepsilon)e^{\frac{-(x-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}} d\varepsilon + \int_{\varepsilon=-\infty}^{\infty} \int_{0}^{t} \frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}} e^{\frac{-(x-\varepsilon)^2}{4D(t-\varepsilon)}} d\tau d\varepsilon \right] e^{rx-st} \quad (4.0)$$

$$\varphi(y,t) = \left[\int_{-\infty}^{\infty} \frac{\emptyset(\varepsilon)e^{\frac{-(y-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}} d\varepsilon + \int_{\varepsilon=-\infty}^{\infty} \int_{0}^{t} \frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}} e^{\frac{-(y-\varepsilon)^2}{4D(t-\varepsilon)}} d\tau d\varepsilon \right] e^{ry-st} \quad (4.1)$$

$$\varphi(z,t) = \left[\int_{-\infty}^{\infty} \frac{\emptyset(\varepsilon)e^{\frac{-(z-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}} d\varepsilon + \int_{\varepsilon=-\infty}^{\infty} \int_{0}^{t} \frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}} e^{\frac{-(z-\varepsilon)^2}{4D(t-\varepsilon)}} d\tau d\varepsilon \right] e^{rz-st} \quad (4.2)$$

Now multiplying the solutions

$$\varphi(r,t) = \left[\int_{-\infty}^{\infty} \frac{\emptyset(\varepsilon)e^{\frac{-(x-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}} d\varepsilon * \int_{-\infty}^{\infty} \frac{\emptyset(\varepsilon)e^{\frac{-(y-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}} d\varepsilon * \int_{-\infty}^{\infty} \frac{\emptyset(\varepsilon)e^{\frac{-(z-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}} d\varepsilon \right] \quad (4.3)$$

$$+ \int_{\varepsilon=-\infty}^{\infty} \int_{0}^{t} \frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}} e^{\frac{-(x-\varepsilon)^2}{4D(t-\varepsilon)}} d\tau d\varepsilon * \int_{\varepsilon=-\infty}^{\infty} \int_{0}^{t} \frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}} e^{\frac{-(y-\varepsilon)^2}{4D(t-\varepsilon)}} d\tau d\varepsilon * \int_{\varepsilon=-\infty}^{\infty} \int_{0}^{t} \frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}} e^{\frac{-(z-\varepsilon)^2}{4D(t-\varepsilon)}} d\tau d\varepsilon \quad (4.4)$$

So and where $r *= xi + yj + zk$

$$\varphi(r,t) = \left[\int_{-\infty}^{\infty} \frac{\emptyset(\varepsilon)\left[e^{\frac{-(x-\varepsilon)^2}{4Dt}+\frac{-(y-\varepsilon)^2}{4Dt}+\frac{-(z-\varepsilon)^2}{4Dt}}\right]}{8\pi^{3/2}\sqrt{D*D*D}(t^{3/2})} d\varepsilon \right] e^{rr*-st} + \quad (4.5)$$

$$+ \left[\int_{\varepsilon=-\infty}^{\infty} \int_{0}^{t} \frac{p(\varepsilon,\tau)}{8\pi^{3/2}\sqrt{D*D*D}(t-\tau)^{3/2}} e^{\frac{-(x-\varepsilon)^2}{4D(t-\varepsilon)}+\frac{-(y-\varepsilon)^2}{4D(t-\varepsilon)}+\frac{-(z-\varepsilon)^2}{4D(t-\varepsilon)}} d\tau d\varepsilon \right] e^{rr*-st}$$

With P (r, t) = 0

$$\varphi(r,t) = \left[\int_{-\infty}^{\infty} \frac{\emptyset(\varepsilon) e^{\frac{-(x-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}} d\varepsilon * \int_{-\infty}^{\infty} \frac{\emptyset(\varepsilon) e^{\frac{-(y-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}} d\varepsilon * \int_{-\infty}^{\infty} \frac{\emptyset(\varepsilon) e^{\frac{-(z-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}} d\varepsilon\right] e^{rr*-st}$$

The solutions are multiplying so add the exponents

$$\boldsymbol{\varphi(r,t)} = \left[\int_{-\infty}^{\infty} \frac{\emptyset(\boldsymbol{\varepsilon}) \left[e^{\frac{-(x-\varepsilon)^2}{4Dt} + \frac{-(y-\varepsilon)^2}{4Dt} + \frac{-(z-\varepsilon)^2}{4Dt}}\right]}{8\pi^{3/2}\sqrt{D*D*D}(t^{3/2})} d\boldsymbol{\varepsilon}\right] e^{rr*-st} \quad \textbf{(4.6)}$$

$$\boldsymbol{\varphi(r,t)} = \left[\int_{-\infty}^{\infty} \frac{\emptyset(\boldsymbol{\varepsilon}) \left[e^{\frac{-(x-\varepsilon)^2}{4Dt} + rx - st \frac{-(y-\varepsilon)^2}{4Dt} + ry - st \frac{-(z-\varepsilon)^2}{4Dt} + rz - st}\right]}{8\pi^{3/2}\sqrt{D*D*D}(t^{3/2})} d\boldsymbol{\varepsilon}\right] \quad \textbf{(4.7)}$$

If the initial condition is (t=0) = 0=$\emptyset\varepsilon$

$$\boldsymbol{\varphi(r,t)} = \left[\int_{\varepsilon=-\infty}^{\infty} \int_0^t \frac{p(\varepsilon,\tau)}{8\pi^{3/2}\sqrt{D*D*D}(t-\tau)^{3/2}} e^{\frac{-(x-\varepsilon)^2}{4D(t-\varepsilon)} + \frac{-(y-\varepsilon)^2}{4D(t-\varepsilon)} + \frac{-(z-\varepsilon)^2}{4D(t-\varepsilon)}} d\tau d\varepsilon\right] e^{rr*-st}$$

Thus we have an expression of the 3D solution using Laplace Transform, this solution is similar to ([11] Analytical Solution of Diffusion Equation for 2D and 3D System)

5. Examples of Solutions by Laplace Transform and Fourier

For these examples were taken shaped cups an atmospheric pollutant, in this case Ozone values of wind speed, and diffusion coefficient of the pollutant chemical reaction with its environment and a point location in specific

Q (t) = 100	**u = 5.87 m / s**
***μ* It is the diffusion coefficient**	***μ*= 0.60 m^2/s**
U is the wind speed	***σ* = 0.1**
***σ* Chemical Reaction coefficient**	**x0 = 0.2 km**

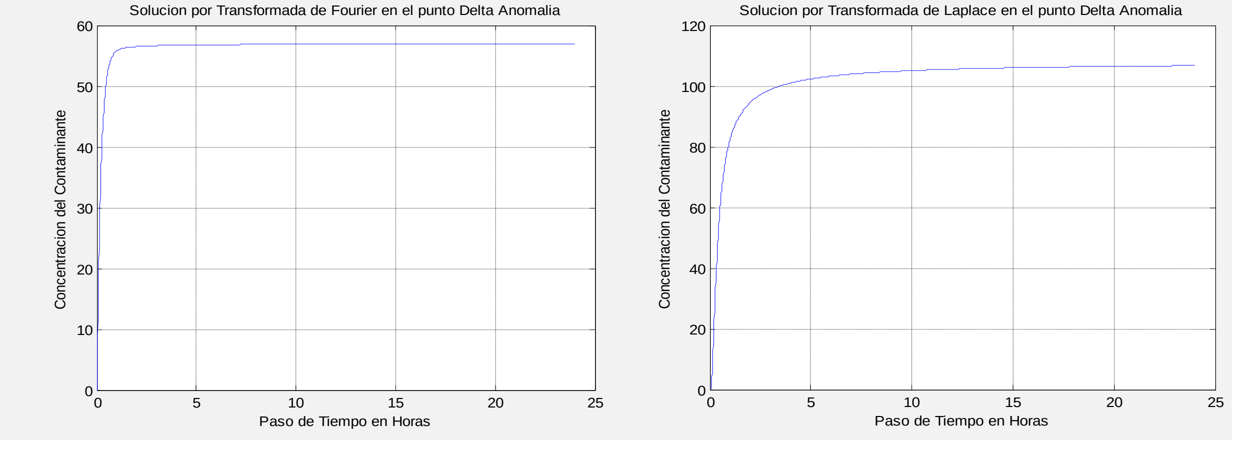


Fig. (1) What we see is decreased contaminant cup by Fourier solution 50% relative to the solution that is softer Laplace graphics.

Q (t) Pulse function

$$\boldsymbol{Q(t)} = \left\{ \begin{matrix} \mathbf{00} < t < 2 \\ \mathbf{1002} < t < 4 \\ \mathbf{04} > t \end{matrix} \right\}$$

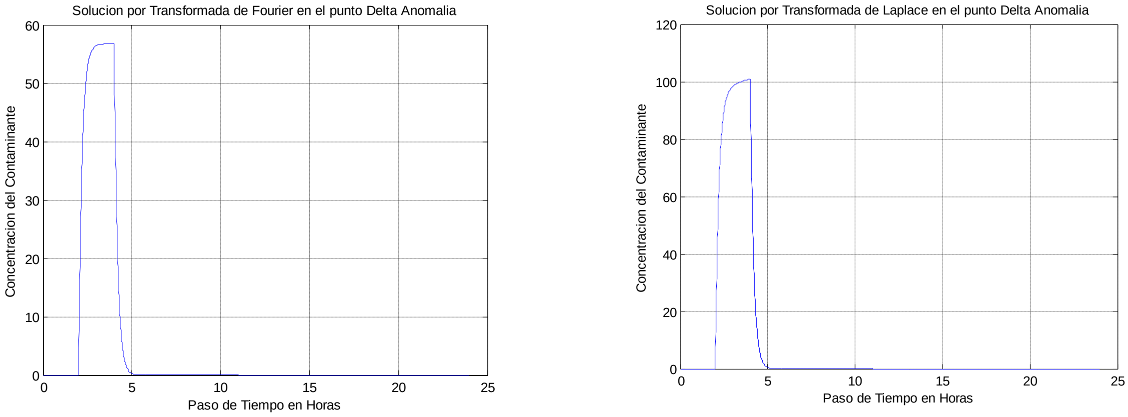


Fig. (2) In this example can be seen following the same effect of the solution on the solution Fourier Laplace and the pulse shape is because advective effect predominates derivative wind drag.

Sinusoid function

$$\boldsymbol{Q(t) = 100t^2 \sin(t)^2 e^{-\frac{t}{2}}}$$

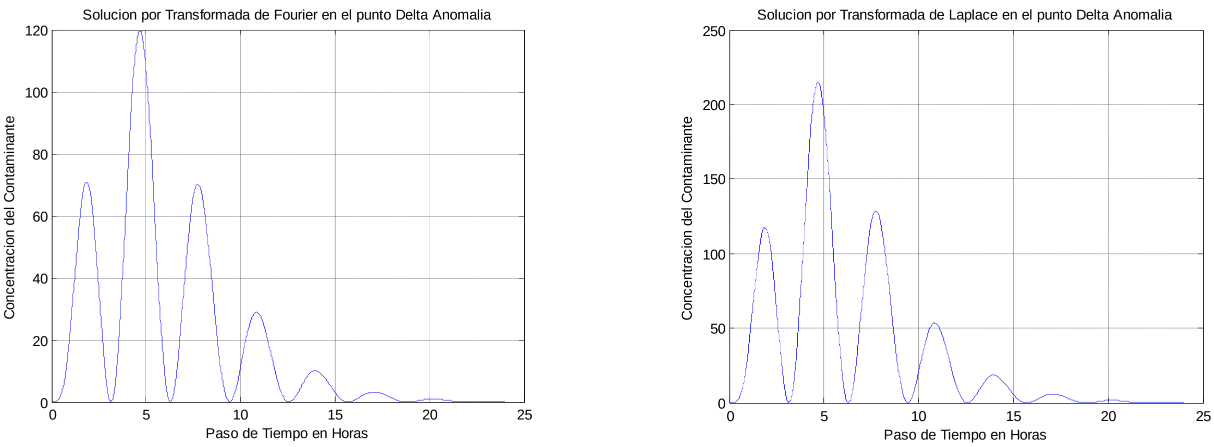


Fig. (3) Seno through the two solutions

Rationalfunction

$$\boldsymbol{Q(t) = \frac{a + bt}{1 + ct + dt^2}}$$

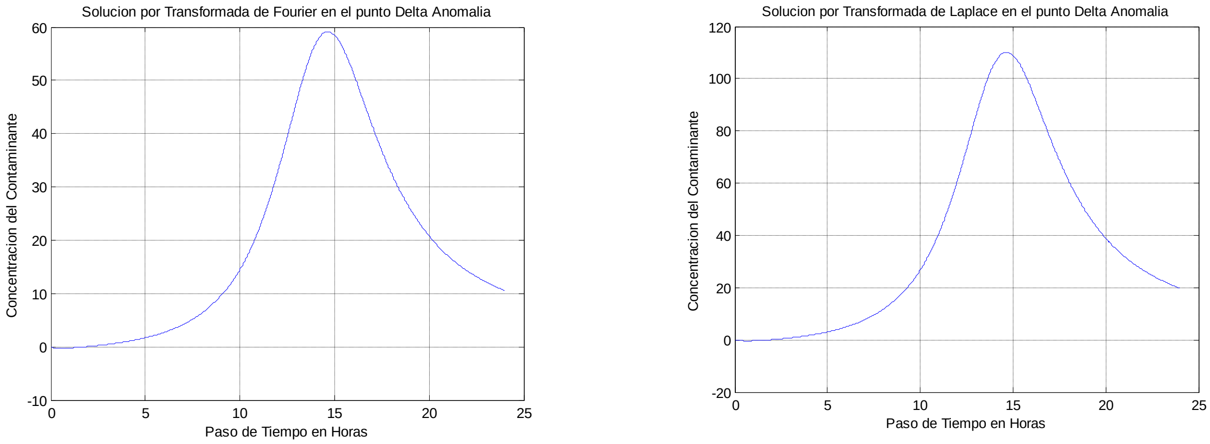


Fig. (4) Rational through the two solutions

Example 2, varying the data with a rate lower wind and most turbulent diffusion which is the predominant Pulse that effect.

Q (t) = 100	**Q = 100**
μIt is the diffusion coefficient	**u = 1.87 m / s**
U is the wind speed	**μ= 0.80 m^2/s**
σ Chemical Reaction coefficient	**σ = 0.3**
	x0 = 0.2 km

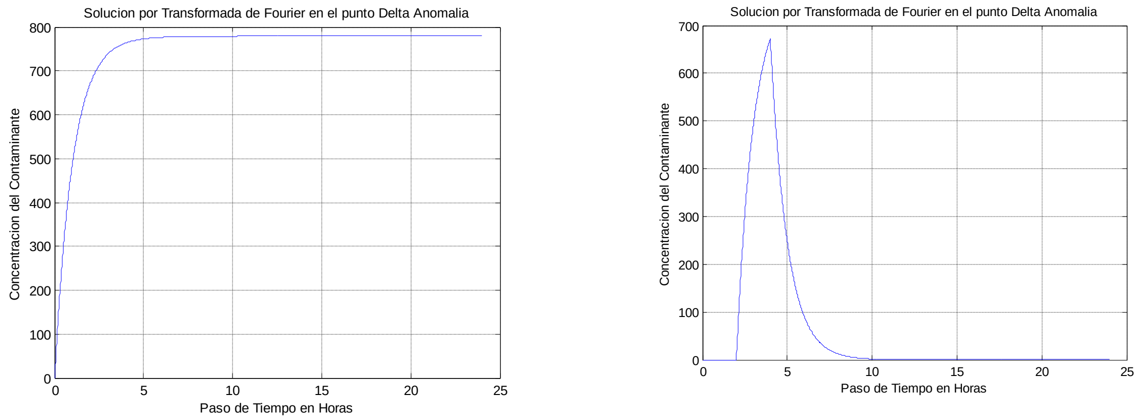
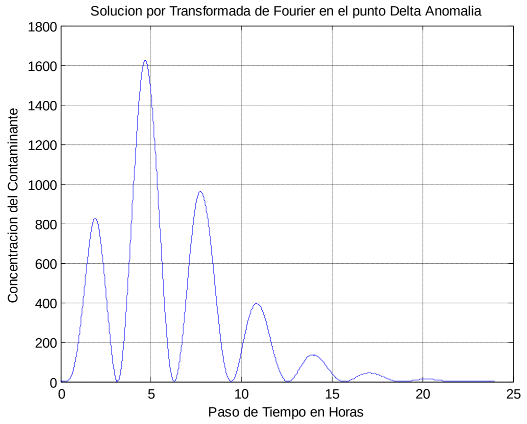


Fig. (5) Constant and Pulse by solving Fourier



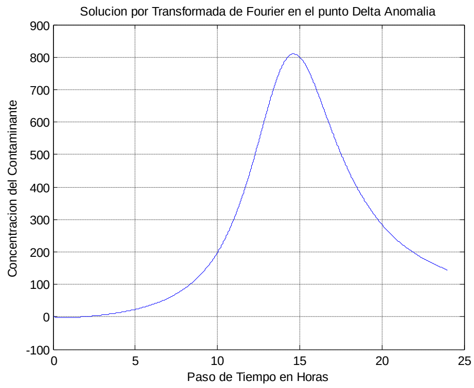


Fig. (6) Seno and rational solution through Fourier

In these graphs we can see that with the solution of Fourier and this input data concentration is too high.

Conclusions

With these solutions found we can conclude that both expressions by both Fourier and Laplace are very similar, except to make changes in the transformation which the Contaminant appears reduced in the Concentration, both solutions retain the initial condition if it is zero, if we reduce The Coefficient of Diffusion and the Wind Speed are also seen the effects raising the Concentration a lot, in the examples, it was not applied with the solution of the Fourier Integral given that it is the same expression as the previous solution, it was also found a 3D expression for this case, which can be evaluated for study, in addition to these 1D models can be studied in the case where to reduce the equation of diffusion Advection Reaction prevails the term of diffusion plus forcing. Another point to consider is that the transformation function will give an increase or decrease in the final solution, the exponential so with large times the effect could be less, and that same term comes in when calculating the expression of the cup to be modeled.
Will now an annex to the Fourier Integral solution which the solution is equal to Laplace, the solution of Laplace - Fourier gives more stability to the dispersion behavior of the contaminant. Some of these solutions are also known but under different conditions and the start of the forcing, but these solutions are found under the transformation reduction.

Annexed

Solution by Fourier Integral

Using the same separation variable with Equation Reduced gives

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} + P(x,t)$$

With

$$\varphi(\pm\infty, t) = 0 \qquad t > 0$$

$$\varphi(x, t0) = \emptyset(x)$$

With t0 = 0

For the homogeneous case, as Eigenvaluesλ

$$\boldsymbol{\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2}} \qquad \textbf{(1)}$$

$$\boldsymbol{X'' + \lambda^2 X = 0}$$
$$\boldsymbol{T' + D\lambda^2 T = 0}$$

Withsolutionslike

$$X(x) = [An\cos(\lambda x) + Bn\sin(\lambda x)] \quad (2)$$

$$T(t) = e^{-D\lambda^2 t}$$

Now with the initial condition must be

$$w(x,t) = \int_{-\infty}^{\infty} [An\cos(\lambda x) + Bn\sin(\lambda x)]e^{-D\lambda^2 t} d\lambda \quad (3)$$

$$w(x,t) = \int_{-\infty}^{\infty} \left[\left(\int_{-\infty}^{\infty} f(\varepsilon)\cos(\lambda\varepsilon)d\varepsilon\right)\cos(\lambda x) + \left(\int_{-\infty}^{\infty} f(\varepsilon)\sin(\lambda\varepsilon)d\varepsilon\right)\sin(\lambda x)\right] e^{-D\lambda^2 t} d\lambda$$

$$w(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\cos(\lambda\varepsilon)\cos(\lambda x) + \sin(\lambda\varepsilon)\sin(\lambda x)]\, f(\varepsilon) e^{-D\lambda^2 t} d\lambda d\varepsilon$$

By trigonometric identities you have to

$$w(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\varepsilon)[\cos(\lambda(\varepsilon - x))]\, e^{-D\lambda^2 t} d\lambda d\varepsilon \quad (4)$$

Let's see parts

$$u = \int_{-\infty}^{\infty} f(\varepsilon)d\varepsilon \; y v = \int_{-\infty}^{\infty} \cos(\lambda(\varepsilon - x))e^{-D\lambda^2 t}\, d\lambda \quad (5)$$

Let us find the integral of the right is integral has the form

$$v(x) = \int_{-\infty}^{\infty} \cos(xp)e^{-p^2}\, dp \quad (6)$$

Differentiating under the integral

$$v'(x) = \int_{-\infty}^{\infty} -p\sin(xp)e^{-p^2}\, dp \quad (7)$$

Now integrating by parts from $-\infty < x < \infty$

$$= \left[\frac{1}{2}\sin(xp)e^{-p^2}\right] - \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{2} x\cos(xp)e^{-p^2}\, dp$$

It is noted that in the above expression

$$[v'(x)] = \frac{x}{2}v(x) \quad (8)$$

$$[v'(x)] - \frac{x}{2}v(x) = 0$$

This is an ODE, so the integrated

$$v(x) = ke^{\frac{x^2}{4}} \quad (9)$$

With $k = e^c$

With

$$k = v(x)e^{\frac{-x^2}{4}}$$

integrating

$$k = \int_0^\infty v(x)e^{\frac{-x^2}{4}}dx$$

A known and v = integral 1

$$k = \int_0^\infty e^{\frac{-x^2}{4}}dx = \int_0^\infty e^{-p^2}dp \tag{10}$$

With

$$p^2 = \frac{x^2}{4} = p = \frac{x}{2}$$

$$k = \frac{\sqrt{\pi}}{2}$$

$$v(x) = \frac{\sqrt{\pi}}{2}e^{-\frac{x^2}{4}}$$

Yes

$$x = \frac{a}{b}$$

$$v(x) = \frac{\sqrt{\pi}}{2}e^{-\frac{a^2}{4b^2}}$$

Thus

$$\int_{-\infty}^{\infty} cos(\lambda(\varepsilon - x))e^{-D\lambda^2 t}\,d\lambda = \sqrt{\frac{\pi}{Dt}}e^{-\frac{(x-\varepsilon)^2}{4Dt}} \tag{11}$$

With

$$\varepsilon = \sqrt{Dt}\lambda a = x - \varepsilon b = \sqrt{Dt}$$

Finally

$$u = \frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} f(\varepsilon)e^{-\frac{(x-\varepsilon)^2}{4Dt}}d\varepsilon$$

$$w(x,t) = \frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} f(\varepsilon)e^{-\frac{(x-\varepsilon)^2}{4Dt}}d\varepsilon$$

$$\varphi(x,t) = \left[\frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} f(\varepsilon)e^{-\frac{(x-\varepsilon)^2}{4Dt}}d\varepsilon\right]e^{rx-st} \tag{12}$$

In the case inhomogeneous

$$\varphi(x,t)=\left[\frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} f(\varepsilon)e^{-\frac{(x-\varepsilon)^2}{4Dt}}d\varepsilon+\int_{\varepsilon=-\infty}^{\infty}\int_{0}^{t}\frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}}e^{\frac{-(x-\varepsilon)^2}{4D(t-\varepsilon)}}d\tau d\varepsilon\right]e^{rx-st} \quad (13)$$

It can be seen that the solution is identical to Laplace, which can be seen in the solution and graphed.

Table.1 Solutions found with

$$\varphi(r,t0)=\varphi^0(r)$$

Fourier Transform

$$\varphi(x,t)=\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega x}\left[\int_0^t e^{-D\omega^2(t-\tau)}\int_{-\infty}^{+\infty}P(\varepsilon,\tau)e^{i\omega\varepsilon}d\varepsilon\right]d\tau d\omega\right)e^{rx}$$

Laplace Transform

$$\varphi(x,t)=\left[\int_{-\infty}^{\infty}\frac{\emptyset(\varepsilon)e^{\frac{-(x-\varepsilon)^2}{4Dt}}}{\sqrt{4\pi Dt}}d\varepsilon+\int_{\varepsilon=-\infty}^{\infty}\int_0^t\frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}}e^{\frac{-(x-\varepsilon)^2}{4D(t-\tau)}}d\tau d\varepsilon\right]e^{rx}$$

By the Fourier integral

$$\varphi(x,t)=\left[\frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty}f(\varepsilon)e^{-\frac{(x-\varepsilon)^2}{4Dt}}d\varepsilon+\int_{\varepsilon=-\infty}^{\infty}\int_0^t\frac{p(\varepsilon,\tau)}{\sqrt{4\pi D(t-\tau)}}e^{\frac{-(x-\varepsilon)^2}{4D(t-\varepsilon)}}d\tau d\varepsilon\right]e^{rx}$$

Laplace Transform - Fourier integral

$$\varphi(r,t)=\left[\int_{-\infty}^{\infty}\frac{\emptyset(\varepsilon)\left[e^{\frac{-(x-\varepsilon)^2}{4Dt}+\frac{-(y-\varepsilon)^2}{4Dt}+\frac{-(z-\varepsilon)^2}{4Dt}}\right]}{8\pi^{3/2}\sqrt{D*D*D}(t^{3/2})}d\varepsilon\right]e^{rr*}+$$

$$+\left[\int_{\varepsilon=-\infty}^{\infty}\int_0^t\frac{p(\varepsilon,\tau)}{8\pi^{3/2}\sqrt{D*D*D}(t-\tau)^{3/2}}e^{\frac{-(x-\varepsilon)^2}{4D(t-\varepsilon)}+\frac{-(y-\varepsilon)^2}{4D(t-\varepsilon)}+\frac{-(z-\varepsilon)^2}{4D(t-\varepsilon)}}d\tau d\varepsilon\right]e^{rr*}$$

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