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General Analytical Solution Unidimensional Advection-Diffusion-Reaction Equation Inhomogeneous on a Bounded Domain with an Application in Dispersion of Pollutants

M.Sc.José Roberto Zenteno Jiménez
Geophysical Engineering, National Polytechnic Institute, Mexico City,
ESIA-Unit Ticoman Delegation Gustavo A. Madero
Email: jzenteno@ipn.mx

Abstract: This paper presents the advection diffusion reaction equation under conditions of general mixed border, for a case of transport of air pollution which is solved by the Fourier series giving an analytical the case of numerical solution alternative and a forcing in specific point, software used for the simulation is Matlab 2016, two examples with two pollutants rates are used, optimum results are showing within the conditions set by the variable speed wind, diffusion coefficient and chemical reaction.

Keywords: Fourier series, advection diffusion reaction equation, variable separation

Introduction

The equation of advection and diffusion is a very important expression given its wide use in the mathematical modeling dispersion processes and broadcasting mass transport, heat or contaminant within a porous medium, atmospheric or river.

A pollutant contained in ambient air is subject to various physical and chemical processes that influence propagation and some of these processes are: transport advection, sedimentation, turbulent diffusion and transformation by various chemical reactions, also due to the complexity of these processes, the dispersion of each substance by fixed or mobile sources has been emitted into the atmosphere is a dimensional and nonstationary phenomenon. In what follows a linear model that takes into account these processes it is formulated, and it is assumed that all the coefficients contained in the respective settings are known.

We'll start with the general model the system of equations **(Guevara-Parra, D. & Skiba, Y. N [2])**. As it's shown in the following.

$$\frac{\partial \emptyset_k}{\partial t} + U.\nabla\emptyset_k + \nabla.\emptyset_k{}^s - \nabla.(\mu\nabla\emptyset_k) - \frac{\partial}{\partial z}\left(\mu_z \frac{\partial \emptyset_k}{\partial z}\right) + \sigma_k\emptyset_k = f_k(r,t) in D x\ (0,T) \quad (1.1)$$

$$\emptyset_k(r,0) = \emptyset_k^0(r) in D \quad (1.2)$$

$$\emptyset_k^s = -v_k^s\emptyset_k e_3 in\ D \quad (1.3)$$

$$\mu_z\frac{\partial \emptyset_k}{\partial z} - U_n\emptyset_k = -v_k^s\emptyset_k in S_H^- \quad (1.4)$$

$$\mu_z\frac{\partial \emptyset_k}{\partial z} = -v_k^s\emptyset_k in S_H^+ \quad (1.5)$$

$$\mu\nabla\emptyset_k.n - U_n\emptyset_k = 0 in S^- \quad (1.6)$$

$$\mu\nabla\emptyset_k.n = 0 in S^+ \quad (1.7)$$

$$\mu^*\nabla\emptyset_k.n = 0 in S_0 \quad (1.8)$$

$$\nabla.U = 0 in D \quad (1.9)$$

Where $\mathbf{U}(\mathbf{r},t) = (u,v,w)$ of note the wind speed in the region $\mathbf{D}$ and it is supposed to meet the continuity equation. Further, $\sigma_k = \sigma_k(\mathbf{r},t) \geq 0$ It is the coefficient of chemical transformation of k-th contaminant species, and the coefficients $\mu = \mu(\mathbf{r},t) > 0$ and denote the diffusion tensor $\mu^* = \mu^*(r,t) > 0$turbulent, ie,

$$\mu = \begin{bmatrix} \mu_x(r,t) & \\ & \mu_y(r,t) \end{bmatrix} \qquad \mu^* = \begin{bmatrix} \mu_x(r,t) & & \\ & \mu_y(r,t) & \\ & & \mu_z(r,t) \end{bmatrix}$$

Respectively, and $f_k(\mathbf{r},t)$ It is formed by forcing emission rates for k-th contaminant:

$$f_k(\mathbf{r},t) = \sum_{i=1}^{N} q_{ik}(t)\delta(\mathbf{r}-\mathbf{r}_i) \tag{1.10}$$

Where $\delta(\mathbf{r}-\mathbf{r}_i)$ is the Dirac delta function centered on the position of the i th point source. Note that the emission rate of each source is the sum of the rates for each pollutant, ie,

$$q_i(t) = \sum_{k=1}^{K} q_{ik}(t) \tag{1.11}$$

The equation defines ϕ_k^0 as the spatial distribution of k-th contaminant species while $t=0$ on $\mathbf{D}$, that is to say, ϕ_k^0 It is the residue k-th contaminant in the atmosphere leaving the industrial activity in a past time interval (for example during the day before). The term $\nabla \cdot \vec{\phi_k^s}$ describes the change in concentration of the observed substance (e.g any type of particles..) per unit time, and due to sedimentation; such process is characterized by the sedimentation velocity constant $v_k^s > 0$.

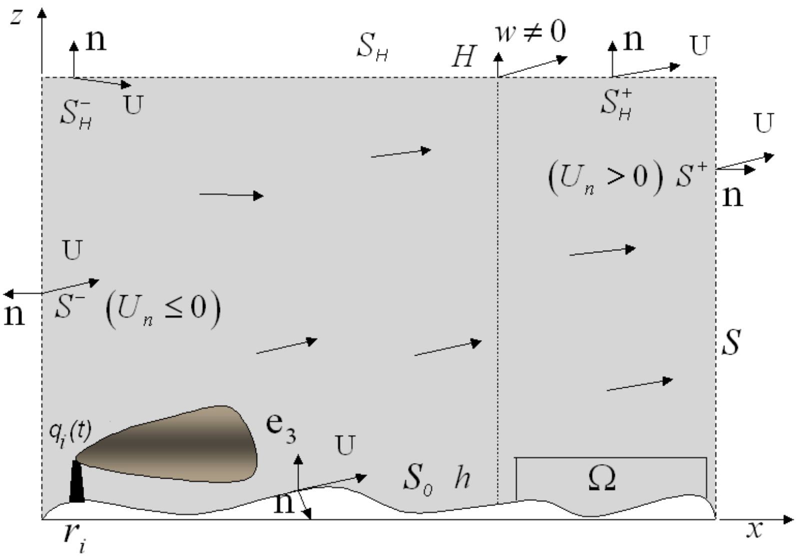


Fig.1 Projected on the XZ plane dispersion of the conceptual model.
(© [2016] [Recovery Emission Rate of Pollutant Source: analysis of existence, uniqueness and stability of the solutions] .Usage with permission).

Border S It was divided into five parts, two for horizontal flow, S^+ It is defined as points S such that $U_n = \mathbf{U}\cdot\vec{n} > 0$, where $\vec{n}$ It is outside normal vector, and S^- It is defined as the complement ($U_n = \mathbf{U}\cdot\vec{n} \leq 0$). Two for the top border: S_H^- it is the part of the border where sedimentation is the result of diffusion minus Advection (equation 1.4), and which implies that sedimentation equals diffusion (equation 1.5). Finally, in, which is the lower boundary, we have that the diffusion is zero because the flow is tangent to the irregular surface (equation 1.8). The boundary condition (1.6) states that when the wind enters the region the total flow of the pollutant, taking into account diffusion and advection, is equal to zero, therefore, there is no outlet or entry of the contaminant species. The boundary condition (1.7) establishes that when the wind leaves the region the turbulent diffusive flow is disregarded in comparison with the Advective flow of the pollutant, therefore, the exit of the polluting species is only by Advection. These boundary conditions were defined by

Marchuk [1], and generalized to the three-dimensional case by **Skiba [6] and Skiba, Y. N. & Parra-Guevara, D. [7].**

To conclude the definition of the previous model, we discuss the meaning of the continuity equation, which closes the dispersion model, which is a condition of incompressibility.

The boundary condition (1.6) states that when the wind enters the region the total flow of the pollutant, taking into account diffusion and advection, is equal to 0, therefore, there is no outlet or entry of the contaminant species.

Now according to these boundary conditions the (1.7) no diffusion despises the output of the pollutant or concentration of the substance but also by advection plus a pollutant concentration that is the case C1

This condition relates Robin inhomogeneous flow to the external source and by the rate C. In this case have a constant, or at least not dependent on the elimination of concentration in the system (or injection if c < 0).$\boldsymbol{\varphi}$

An example might be as pollution, c acting as eliminating contamination at the boundary from inside to outside the domain, regardless of the amount already in the domain.$\boldsymbol{\varphi}$

Or, if c <0, so the flow is positive in the external address would be an external sourceConcentration from outside the boundary to within the domain.

Now the case to be solved is the following

$$\frac{\partial \emptyset_k}{\partial t} + U.\nabla\emptyset_k - \nabla.(\mu\nabla\emptyset_k) - \frac{\partial}{\partial z}\left(\mu_z \frac{\partial \emptyset_k}{\partial z}\right) + \sigma_k\emptyset_k = f_k(r,t) \quad \textbf{(1.1)}$$

$$\emptyset_k(r,0) = \emptyset_k^0(r) in D \quad \textbf{(1.2)}$$

$$\mu\nabla\emptyset_k.n - U_n\emptyset_k = C1 in S^- \quad \textbf{(1.6)}$$

$$\mu\nabla\emptyset_k.n - U_n\emptyset_k = C2 in S^+ \quad \textbf{(1.7)}$$

$$\nabla.U = 0 in D \quad \textbf{(1.9)}$$

Unidimensional cases of governing equations.

$$\frac{\partial\varphi}{\partial t} + u\frac{\partial\varphi}{\partial x} - \mu\frac{\partial^2\varphi}{\partial x^2} + \sigma\varphi = q(t)\delta(x - x_0) \quad \textbf{(1.12)}$$

$$\varphi(x,0) = \varphi_0 0 < x < l \quad \textbf{(1.13)}$$

$$\mu\frac{\partial\varphi(l,t)}{\partial x} - u\varphi(l,t) = C1 t > 0 \quad \textbf{(1.14)}$$

$$\mu\frac{\partial\varphi(0,t)}{\partial x} - u\varphi(0,t) = C2 \sigma > 0 \quad \textbf{(1.15)}$$

Other directions within the domain of the three-dimensional in the space D

$$\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial y} - \mu\frac{\partial^2\varphi}{\partial y^2} + \sigma\varphi = q(t)\delta(y - y_0) \quad \textbf{(1.16)}$$

$$\varphi(y,0) = \varphi_0 0 < y < l \quad \textbf{(1.17)}$$

$$\mu\frac{\partial\varphi(l,t)}{\partial y} - v\varphi(l,t) = C1 t > 0 \quad \textbf{(1.18)}$$

$$\mu\frac{\partial\varphi(0,t)}{\partial y} - v\varphi(0,t) = C2 \sigma > 0 \quad \textbf{(1.19)}$$

In the case in the z direction it is very similar to the previous case

$$\frac{\partial\varphi}{\partial t} + (w - V_k)\frac{\partial\varphi}{\partial z} - (\mu_z)\frac{\partial^2\varphi}{\partial z^2} + \sigma\varphi = q(t)\delta(z - z_0) \quad \textbf{(1.20)}$$

$$\varphi(z,0) = \varphi_0 0 < z < l \quad \textbf{(1.21)}$$

$$\frac{\partial\varphi(0,t)}{\partial x}+\frac{V_k}{\mu_z}\varphi(0,t)=0t>0 \quad (1.22)$$

$$\frac{\partial\varphi(l,t)}{\partial x}-\frac{2w}{\mu_z}\varphi(l,t)=0\sigma>0 \quad (1.23)$$

Solving the one-dimensional case in the x direction with the following transformation to reduce, where H is the inhomogeneous part.

$$\varphi(x,t)=w(x,t)e^{rx-st}+e^{rx}H(x,t) \quad (1.24)$$

When applying the Transformation we obtain the following:

$$r=\frac{v}{2D} \qquad s=(\frac{v^2}{4D}+\sigma)$$

And the reduced Equation is:

$$\frac{\partial w}{\partial t}=D\frac{\partial^2 w}{\partial x^2}+e^{st}F(x,t) \quad (1.25)$$

$$F(x,t)=\gamma e^{-rx}-\frac{\partial H}{\partial t}+D\frac{\partial^2 H}{\partial x^2}$$

Applying to the Border Conditions For both terms the Homogeneous and Non-homogeneous

$$w(x,t0)=\emptyset(x)e^{-rx+st0}-e^{st0}H(x,t0)$$

So we have

$$w_x(x,t)-rw(x,t)=\frac{vC2}{D}$$

$$H_x(x,t)-rH(x,t)=\frac{vC1}{D}e^{-rx}$$

Homogeneous part

$$w_x(0,t)-rw(0,t)=0$$

$$w_x(l,t)-rw(l,t)=0$$

Non-Homogeneous part

$$H_x(0,t)+rH(0,t)=\frac{vC2}{D}$$

$$H_x(l,t)+rH(l,t)=\frac{vC1}{D}e^{-rl}$$

We have to solve the non-homogenous equation, so let's see the obtaining of this function, which the solution is

$$H(x,l)=\frac{v}{rD}\left[C2\left(e^{-rl}-1\right)+C1\left(1-e^{-rl}\right)\right] \quad (1.26)$$

Accommodating

$$H(x,l)=\left[C2\left(e^{-rl}-1\right)+C1\left(1-e^{-rl}\right)e^{-rl}\right]$$

Now this expression must be in terms of more manageable and harmonic functions, this is the non-homogeneous solution, now let's see this observation

$$\varphi(x,t)=w(x,t)e^{rx-st}+e^{rx}H(x,t)$$

Now

$$\varphi(x,0)=w(x,0)e^{rx}+e^{rx}H(x,0)$$

$$0 = w(x,0) + H(x,0)$$
$$w(x,0) = 0$$
$$H(x,0) = 0$$

By making the matching case homogeneous and also for the non-homogeneous case the sum of both solutions but both can be matched and so we have this

$$\left[(e^{-rl} - 1) + (1 - e^{-rl})e^{-rl}\right] = \cos(x\lambda) + \mathrm{sen}(x\lambda)$$

Which by accommodating the expression we have the following

$$H(x,t) = \left(1 + \cos\left(\frac{\pi x}{l}\right)\right)C2 + \left(1 - \cos\left(\frac{\pi x}{l}\right)\right)e^{-xl}C1 \qquad (1.27)$$

Which is equivalent to

$$H(x,t) = \left[(e^{-rl} - 1)C2 + (1 - e^{-rl})e^{-rl}C1\right]$$

Now a second solution would be the following. Substituting a linear form, where the unknown is the H function

$$H1(x,t) = H(x,t) + P(x,t)$$

Where the function H must verify the boundary conditions, the excess does not matter how the function is obtained, or if it can represent the solution of the problem in large times when applying H1 to the next system it is necessary to

$$F(x,t) = \gamma e^{-rx} - \frac{\partial H}{\partial t} + D\frac{\partial^2 H}{\partial x^2}$$

$$H_x(0,t) + rH(0,t) = \frac{vC2}{D}$$
$$H_x(l,t) + rH(l,t) = \frac{vC1}{D}e^{-rl}$$

$$H1(x,0) = H(x,0) + P(x,0)$$

$$H1(x,0) - H(x,0) = P(x,0)$$

With H1= $\emptyset(x)$

$$P(x,0) = \emptyset(x) - H(x,0)$$

Remaining the system as

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} + \gamma e^{-rx}$$

$$P_x(0,t) + rP(0,t) = 0$$
$$P_x(l,t) + rP(l,t) = 0$$

$$P(x,0) = \emptyset(x) - H(x,0)$$

This form of H is if we choose quadratically the solution

$$H(x,t) = \left[\frac{\frac{vC1}{D}e^{-rl} - \frac{vC2}{D}(1 - rL)}{2L + rL^2}\right]x^2 + \frac{vC2}{D}x \qquad (1.28)$$

Now solving the EDP of the homogeneous part

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2}$$

$$w(x,t0) = \emptyset(x)$$

$$w_x(0,t) - rw(0,t) = 0$$

$$w_x(l,t) + rw(l,t) = 0$$

Having Transcendent as

$$Tan(l\lambda) = -\frac{4\mu u\lambda}{u^2 - 4\mu^2\lambda^2}$$

The General Solution in T is:

$$T(t) = \frac{e^{-D\lambda t}}{\int_0^l \phi(x)^2 dx}\left(\int_0^t e^{(s+D\lambda^2)t}\int_0^l F(x,\zeta)\phi(x)dxd\zeta + e^{(s+D\lambda)t0}\int_0^l (e^{-rx}\emptyset(x) - H(x,t))\phi(x)dx\right) \quad (1.29)$$

$$\varphi(x,t) = w(x,t)e^{rx-st} + e^{rx}H(x,t) \quad 0 < x < l$$

$$\varphi(x,0) = \emptyset(x) = w(x,t0)e^{rx-st} + e^{rx}H(x,t0) \quad 0 < x < l$$

$$\emptyset(x)e^{-rx+st} - e^{rx}H(x,t0) = w(x,t0)$$

Now

$$\emptyset(x)e^{-rx+st} - e^{rx}H(x,t0) = \sum_{n=M}^{\infty}\phi(x)T(t)$$

$$H(x,t0) = \left(1 + \cos\left(\frac{\pi x}{l}\right)\right)C2 + \left(1 - \cos\left(\frac{\pi x}{l}\right)\right)e^{-xl}C1$$

Solving and algebraing a bit gives us the Solution required for an initial condition other than zero and with C. Border also different from zero

$$\varphi(x,t) = \sum_{n=M}^{00}\phi(x)\frac{e^{rx-(s+D\lambda^2)t}}{\int_0^l \phi(x)^2 dx}\left(\int_0^t e^{(s+D\lambda^2)\zeta}\int_0^l e^{-rx}F(x,\zeta)\phi(x)dxd\zeta\right)dxd\zeta + e^{rx}H(x,t) \quad (1.30)$$

$$H(x,t) = e^{(s+D\lambda^2)t0}\int_0^l (e^{-rx}\emptyset(x) - H(x,t0))\phi(x)dx$$

$$H(x,t0) = \left(1 + \cos\left(\frac{\pi x}{l}\right)\right)C2 + \left(1 - \cos\left(\frac{\pi x}{l}\right)\right)e^{-xl}C1$$

$$F(x,t) = \gamma$$

$$H(x,t0) = \left[\frac{\frac{vC1}{D}e^{-rl} - \frac{vC2}{D}(1 - rl)}{2l + rl^2}\right]x^2 + \frac{vC2}{D}x$$

Coefficients with the Modified Delta, this coefficient was demonstrated in (Zenteno Jiménez J.R [3]).

$$dk = \left[\frac{0.5 + \cos(x0\lambda)\left(0.5\left(1 + \frac{\sin(2\lambda)}{2\lambda}\right)\right) + \sin(x0\lambda)\left(\frac{\sin(\lambda)^2}{2\lambda}\right)}{0.5\left(1 + \frac{\sin(2\lambda)}{2\lambda}\right)} - \frac{0.5 + \cos(x0\lambda)\left(\frac{\sin(\lambda)^2}{2\lambda}\right) + \sin(x0\lambda)\left(0.5\left(1 - \frac{\sin(2\lambda)}{2\lambda}\right)\right)}{\frac{r}{2\lambda}\left(1 - \frac{\sin(2\lambda)}{2\lambda}\right)}\right]$$

Solution with the Modified Delta with the initial condition other than zero is;

$$\boldsymbol{\varphi(x,t) = \sum_{n=m}^{k}\left[\cos(x\lambda) + \frac{r}{\lambda}\operatorname{sen}(x\lambda)\right] e^{-(\mu\lambda^2+s)t+rx}\int_0^t d_k e^{(\mu\lambda^2+s)\tau} Q(\tau)d\tau + e^{rx}H(x,t)} \quad \textbf{(1.31)}$$

$$H(x,t) = \int_0^l (e^{-rx}\emptyset(x) - H(x,0))\phi(x)dx$$

$$H(x,t0) = \left(1 + \cos\left(\frac{\pi x}{l}\right)\right)C2 + \left(1 - \cos\left(\frac{\pi x}{l}\right)\right)e^{-xl}C1$$

$$dk = \left[\frac{0.5 + \cos(x0\lambda)\left(0.5\left(1 + \frac{\sin(2\lambda)}{2\lambda}\right)\right) + \sin(x0\lambda)\left(\frac{\sin(\lambda)^2}{2\lambda}\right)}{0.5\left(1 + \frac{\sin(2\lambda)}{2\lambda}\right)} - \frac{0.5 + \cos(x0\lambda)\left(\frac{\sin(\lambda)^2}{2\lambda}\right) + \sin(x0\lambda)\left(0.5\left(1 - \frac{\sin(2\lambda)}{2\lambda}\right)\right)}{\frac{r}{2\lambda}\left(1 - \frac{\sin(2\lambda)}{2\lambda}\right)}\right]$$

Solving the Integrals by parts we obtain the following expressions$\boldsymbol{H(x,t)}$

$$\boldsymbol{I1 = (C2 + C1)\int_0^l e^{-lx}\cos(x\lambda)\,dx + \frac{r}{\lambda}(C2 + C1)\int_0^l e^{-lx}\operatorname{sen}(x\lambda)\,dx} \quad \textbf{(1.32)}$$

$$\boldsymbol{I2 = (C2 - C1)\int_0^l e^{-lx}\cos\left(\frac{x\pi}{l}\right)\cos(x\lambda)\,dx + \frac{r}{\lambda}(C2 - C1)\int_0^l e^{-lx}\cos\left(\frac{x\pi}{l}\right)\operatorname{sen}(x\lambda)\,dx} \quad \textbf{(1.33)}$$

$$\boldsymbol{I3 = \int_0^l e^{-rx}\emptyset(x)\cos(x\lambda)\,dx + \frac{r}{\lambda}\int_0^l e^{-rx}\emptyset(x)\operatorname{sen}(x\lambda)\,dx} \quad \textbf{(1.34)}$$

$$\int_0^l e^{-rx}\emptyset(x)\cos(x\lambda)\,dx = \emptyset(x)\left(\frac{e^{-lx}(\lambda sen(\lambda l) - rcos(\lambda l) + re^{lr})}{\lambda^2 + r^2}\right)$$

$$\frac{r}{\lambda}\int_0^l e^{-rx}\emptyset(x)\operatorname{sen}(x\lambda)\,dx = -\emptyset(x)\left(\frac{re^{-lx}(rsen(\lambda l) + \lambda cos(\lambda l) - \lambda e^{lr})}{\lambda(\lambda^2 + r^2)}\right)$$

$$\int_0^l e^{-lx}\cos(x\lambda)\,dx = (C2 + C1)\left(\frac{e^{-l^2}(\lambda sen(\lambda l) - lcos(\lambda l) + le^{-l^2})}{\lambda^2 + l^2}\right)$$

$$\int_0^l e^{-lx}\,\mathrm{sen}(x\lambda)\,dx = -\frac{r}{\lambda}(C2+C1)\left(\left(\frac{e^{-l^2}\left(lsen(\lambda l)+\lambda cos(\lambda l)-le^{-l^2}\right)}{\lambda^2+l^2}\right)\right)$$

$$\int_0^l e^{-lx}\cos\left(\frac{x\pi}{l}\right)\cos(x\lambda)\,dx$$
$$= -(C2 - C1)\left(\frac{l^2e^{-l^2}(l^2\lambda^3+(l^4-\pi^2)\lambda)sen(\lambda l)+(-l^3\lambda^2-l^5-\pi^2 l)cos(\lambda l)-l^3e^{-l^2}\lambda^2+(-l^5-\pi^2 l)e^{-l^2}}{(l^4\lambda^4+(2l^6-2\pi l^2)\lambda^2+l^8+2\pi^2 l^4+\pi^4)}\right)$$

$$\int_0^l e^{-lx}\cos\left(\frac{x\pi}{l}\right)\mathrm{sen}(x\lambda)\,dx$$
$$= (C2-C1)\left(\frac{l^2e^{-l^2}r\left(l^3\lambda^2+(l^5-\pi^2 l)\right)sen(\lambda l)+(l^2\lambda^3+(l^4-\pi^2)\lambda)cos(\lambda l)+l^2e^{l^2}\lambda^3+(l^4-\pi^2)e^{l^2}\lambda}{\lambda(l^4\lambda^4+(2l^6-2\pi l^2)\lambda^2+l^8+2\pi^2 l^4+\pi^4)}\right)$$

We can make some observations the Eigenvalor Lambda will continue to grow therefore in the parts where we have the breast will be a small term almost to zero, we can omit it and so we will have

$$e^{rx}H(x,t) = \emptyset(x)\left(\frac{e^{-lx}(\lambda sen(\lambda l)-rcos(\lambda l)+re^{lr})}{\lambda^2+r^2}\right)+(C2+C1)\left(\frac{e^{-l^2}\left(\lambda sen(\lambda l)-lcos(\lambda l)+le^{-l^2}\right)}{\lambda^2+l^2}\right)$$

$$e^{rx}H(x,t) = \emptyset(x)\left(\frac{e^{-lx}(\lambda sen(\lambda l)-rcos(\lambda l)+re^{lr})}{\lambda^2+r^2}\right)+(C2+C1)\left(\frac{e^{-l^2}\left(\lambda sen(\lambda l)-lcos(\lambda l)+le^{-l^2}\right)}{\lambda^2+l^2}\right)$$

$$\boldsymbol{H(x,t) = \emptyset(x)\left(\frac{e^{(r-l)x}\left(\lambda sen(\lambda l)-rcos(\lambda l)+re^{lr}\right)}{\lambda^2+r^2}\right)+(C2+C1)\left(\frac{e^{(rx-l^2)}\left(\lambda sen(\lambda l)-lcos(\lambda l)+le^{-l^2}\right)}{\lambda^2+l^2}\right)} \tag{1.35}$$
$$\boldsymbol{-(C2-C1)e^{rx}\left(\frac{l^2e^{-l^2}(l^2\lambda^3+(l^4-\pi^2)\lambda)sen(\lambda l)+\left(-l^3\lambda^2-l^5-\pi^2 l\right)cos(\lambda l)-l^3e^{-l^2}\lambda^2+\left(-l^5-\pi^2 l\right)e^{-l^2}}{(l^4\lambda^4+(2l^6-2\pi l^2)\lambda^2+l^8+2\pi^2 l^4+\pi^4)}\right)}$$

Solution with the Modified Delta with the initial condition other than zero is the following, omitting the third integral given that the Eigenvalue continues to increase, we have the solution for an address in this case x

$$\varphi(x,t) = \sum_{n=m}^{k}\left[\cos(x\lambda)+\frac{r}{\lambda}\mathrm{sen}(x\lambda)\right]e^{-(\mu\lambda^2+s)t+rx}\int_0^t d_k e^{(\mu\lambda^2+s)\tau}Q(\tau)d\tau + H(x,t)$$

$$H(x,t) = \emptyset(x)\left(\frac{e^{(r-l)x}(\lambda sen(\lambda l)-rcos(\lambda l)+re^{lr})}{\lambda^2+r^2}\right) + (C2+C1)\left(\frac{e^{(rx-l^2)}\left(\lambda sen(\lambda l)-lcos(\lambda l)+le^{-l^2}\right)}{\lambda^2+l^2}\right)$$

Applications to other similar physical processes are

Solute transport in Aquifers hereφIt is the porosity and the concentration C, where the density of the matrix and solute S is absorbed on the pore surface$\boldsymbol{\rho}$

$$\boldsymbol{\frac{\partial(C\varphi)}{\partial t} - V\nabla C = \nabla.\left(D(C\varphi)\right)+F} \tag{1.36}$$

$$\boldsymbol{F = -\frac{\partial\rho S}{\partial t} con K = S, F = -K\rho\frac{\partial C}{\partial t}}$$

In one direction is

$$\boldsymbol{R\frac{\partial C}{\partial t} = -v\frac{\partial C}{\partial x} + D\frac{\partial^2 C}{\partial x^2}}$$

$$R = \left(1 + \frac{K\rho}{\varphi}\right)$$

$$C(x, 0) = 0 \quad 0 < x < l$$

$$D\frac{\partial C(l,t)}{\partial x} - vC(l,t) = 0 \quad t > 0$$

$$D\frac{\partial C(0,t)}{\partial x} - vC(0,t) = C0 \quad \sigma > 0$$

Groundwater movement in porous media

$$S\frac{\partial h}{\partial t} = \nabla.\left(Dh(\nabla h)\right) + N \tag{1.37}$$

Where S is the coefficient storage Recharging and N

In one direction it is the next but is not linear

$$S\frac{\partial h}{\partial t} = D\left[\frac{d}{dx}\left(h\frac{\partial h}{\partial x}\right) - \alpha\frac{\partial h}{\partial x}\right] + N(t)$$

$$y = h^2 now$$

$$\frac{\partial^2 y}{\partial x^2} - 2\alpha\frac{\partial y}{\partial x} + \frac{2N(t)}{D} = \frac{1}{k}\frac{\partial y}{\partial t}$$

$$a = \frac{\alpha}{2D} \quad k = \frac{Dd}{S}$$

$$N(t) = N(t)\delta(x - x0)$$

$$y(x, 0) = 0 \quad 0 < x < l$$

$$D\frac{\partial y(l,t)}{\partial x} - vy(l,t) = 0 \quad t > 0$$

$$D\frac{\partial y(0,t)}{\partial x} - vy(0,t) = 0 \quad \sigma > 0$$

Equations for phase flow through a porous medium homogeneous, for slightly compressible flow

$$\frac{D\varphi}{k}\frac{\partial p}{\partial t} = \nabla.\left((\nabla p)\right) + \frac{D}{\rho k}I \tag{1.38}$$

Where D is the diffusion coefficient constant k, I source $\rho density and \varphi porosity$

In an address it is as follows

$$\frac{\partial P}{\partial t} = \frac{k}{D\varphi}\frac{\partial^2 P}{\partial x^2} + \frac{1}{\rho\varphi}I$$

$$I = I(t)\delta(x - x0)$$

$$P(x, 0) = 0 \quad 0 < x < l$$

$$D\frac{\partial P(l,t)}{\partial x} - vP(l,t) = 0t > 0$$
$$D\frac{\partial P(0,t)}{\partial x} - vP(0,t) = 0\sigma > 0$$

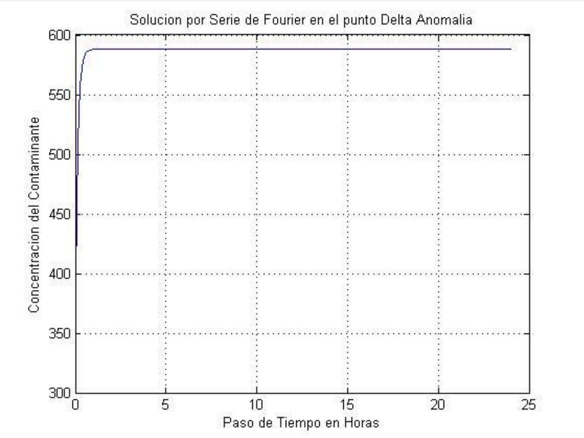
Now we will apply some examples with the solution found

The following tests are found with the solution

Examples 1

Q (t) = 100	**u = 5.87 m / s**
μ It is the diffusion coefficient	**μ = 0.60 m^2/s**
U is the wind speed	**σ = 0.1**
σ Chemical Reaction coefficient	**x0 = 0.2 km**
∅ initial mass	**∅ = 50**
C1 and C2	**C1 = 10**
	C2 = 10

Q (t) = 100



Q (t) Pulso Function

$$Q(t) = \begin{cases} 0 & 0 < t < 2 \\ 100 & 2 < t < 4 \\ 0 & 4 > t \end{cases}$$

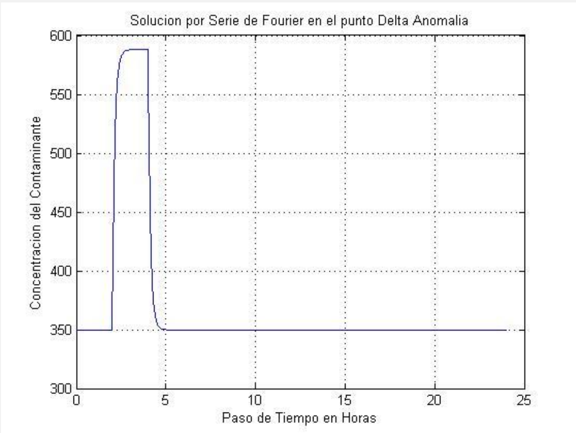
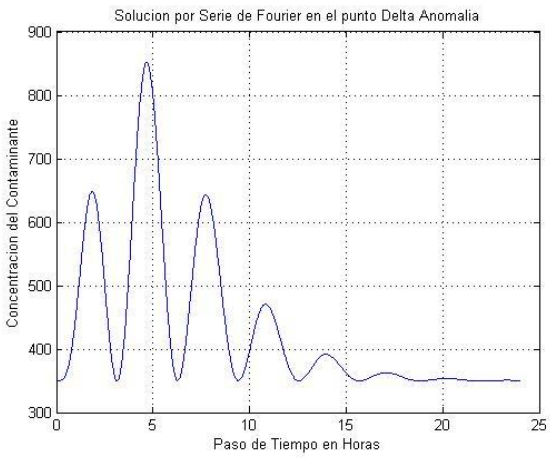


Figure.2 we can see that the modeled function starts above the initial value given

Sinusoid function

$$Q(t) = 100t^2 \sin(t)^2 e^{-\frac{t}{2}}$$



Rational function

$$Q(t) = \frac{a + bt}{1 + ct + dt^2}$$

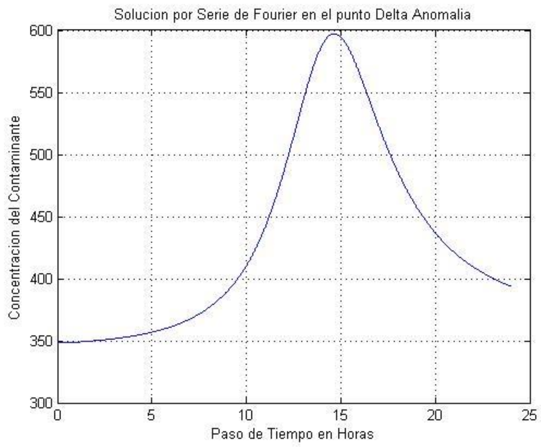


Figure.3 we can observe the same behavior as the previous case

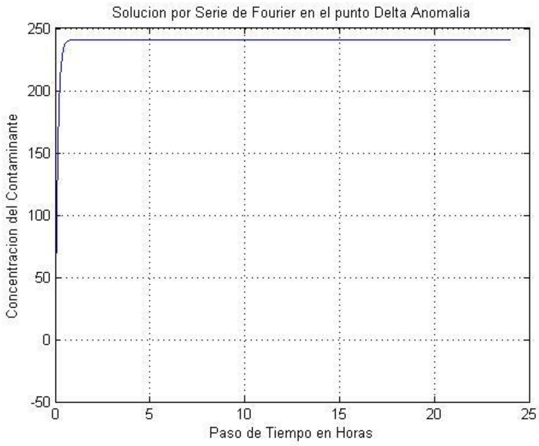
Examples 2

Q (t) = 100	**Q = 100**
μIt is the diffusion coefficient	**u = 3.87 m / s**
U is the wind speed	**μ= $0.50 m^2/s$**
σ Chemical Reaction coefficient	**σ** = 0.1
∅ initial mass	x0 = 0.2 km
C1 and C2	**$\emptyset = 20$**
	C1 = 30
	C2 = 10

Q (t) = 100

Q (t) Pulse function

$$Q(t) = \begin{cases} 00 < t < 2 \\ 1002 < t < 4 \\ 04 > t \end{cases}$$



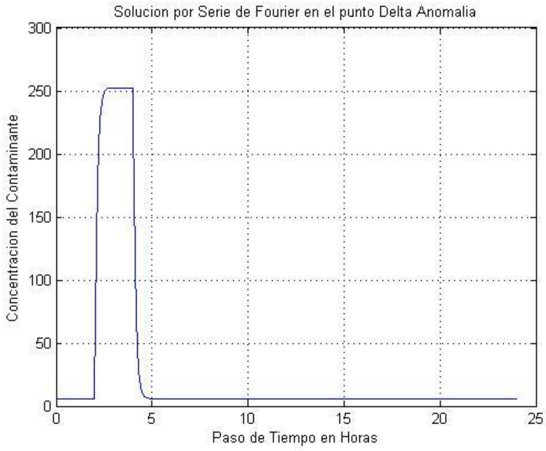


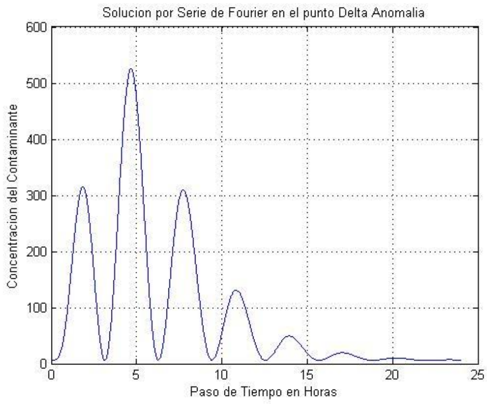
Figure.4 In these cases the advective process is dominant

Sinusoid function

$$Q(t) = 100t^2 \sin(t)^2 e^{-\frac{t}{2}}$$

Rational function

$$Q(t) = \frac{a + bt}{1 + ct + dt^2}$$



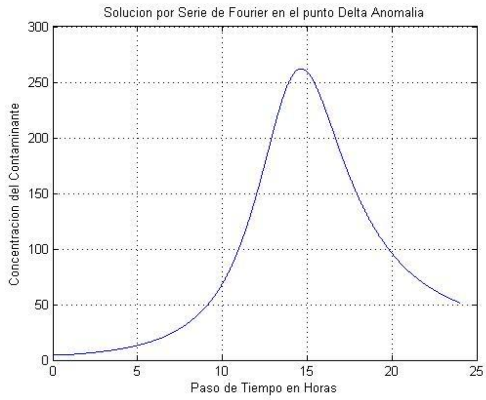


Figure.5 In these examples you can see that the start is different

Now using other following transformation proposal, we can see that the solution to this equation is of this type$\varphi(x,t) = X(x,t) * Y(x,t)$ **so we propose follows that one of them is exponentially being X and Y**

so $\boldsymbol{\varphi(x,t) = Z(x,t) * X(x,t) + J(x,t) * Y(x,t)}$ to find the solution having the inhomogeneous part and also a way to obtain all the transformation functions used for this case and where all that changes is the sign of the coefficients.

$$\boldsymbol{\varphi(x,t) = Z(x,t)e^{\alpha x - \beta t} + J(x,t)e^{\alpha 1 x + \beta 1 t}} \quad \textbf{(1.39)}$$

Making substitution within of (1.12), we have

$$\frac{\partial z}{\partial t} = D\frac{\partial^2 z}{\partial x^2} + (2\alpha D - u)\frac{\partial z}{\partial x} + (\alpha^2 D - u\alpha + \beta - \sigma)z + fe^{\beta t - \alpha x}$$
$$+\left(-\frac{\partial J}{\partial t} + D\frac{\partial^2 J}{\partial x^2} + (2\alpha 1 D - u)\frac{\partial J}{\partial x} + (\alpha 1^2 D - u\alpha 1 - \beta 1 - \sigma)J\right)e^{\beta 1 t + \alpha 1 x + \beta t - \alpha x}$$

With these two factors to Z

$$\boldsymbol{\alpha = \frac{u}{2D} y \beta = \frac{u^2}{4D} + \sigma}$$

And for J

$$\boldsymbol{\alpha 1 = \frac{u}{2D} y \beta 1 = -\left(\frac{u^2}{4D} + \sigma\right)}$$

Accommodating and recalling that J is the inhomogeneous part

$$\frac{\partial z}{\partial t} = D\frac{\partial^2 z}{\partial x^2} + fe^{\beta t - \alpha x} + \left[\left(-\frac{\partial J}{\partial t} + D\frac{\partial^2 J}{\partial x^2}\right)e^{\beta 1 t}\right]$$
$$\frac{\partial z}{\partial t} = D\frac{\partial^2 z}{\partial x^2} + e^{\beta t}\left[fe^{-\alpha x} + \left(-\frac{\partial J}{\partial t} + D\frac{\partial^2 J}{\partial x^2}\right)\right]$$
$$\frac{\partial z}{\partial t} = D\frac{\partial^2 z}{\partial x^2} + e^{\beta t}\left[e^{-\alpha x}\left(f + \left(-\frac{\partial J}{\partial t} + D\frac{\partial^2 J}{\partial x^2}\right)e^{\alpha x}\right)\right]$$
$$\frac{\partial z}{\partial t} = D\frac{\partial^2 z}{\partial x^2} + e^{\beta t}\left[e^{-\alpha x}(J(x,t))\right]$$

Now the part of Z is the homogeneous condition and the boundary conditions like while part J is inhomogeneous in the boundary conditions, remaining in the final solution as

$$\frac{\partial z}{\partial t} = D\frac{\partial^2 z}{\partial x^2} + e^{\beta t}F(x,t) \quad \textbf{(1.40)}$$

$$F(x,t) = \gamma e^{-\alpha x} - \frac{\partial J}{\partial t} + D\frac{\partial^2 J}{\partial x^2}$$

In the case Z

$$F(x,t) = \gamma e^{-\alpha x}$$

Where gamma is

$$\gamma = q(t)\delta(x - x_0) \quad \textbf{(1.41)}$$

Applying the boundary conditions have:

$$z(x,0) = 0 \quad \textbf{(1.42)}$$
$$z_x(0,t) - \alpha z(0,t) = 0 \quad \textbf{(1.43)}$$
$$z_x(l,t) + \alpha z(l,t) = 0 \quad \textbf{(1.44)}$$

Note: transforming the boundary condition

$$\mu\frac{\partial \varphi(l,t)}{\partial x} = 0 \qquad t > 0 \quad \textbf{(1.45)}$$

With z (x,) the same transformation is obtainedt_0

$$z_x(l,t) + \alpha z(l,t) = 0 \quad \textbf{(1.46)}$$

Recalling the solution to the above case we have the same solution found

$$\varphi(x,t) = \sum_{n=1}^{\infty} \phi_n(x) \frac{e^{\alpha x-(\beta+D\lambda_n{}^2)t}}{\int_0^l \phi_n(x)^2 dx} \left(\int_0^t e^{(\beta+D\lambda_n{}^2)\zeta} \int_0^l F(x,\zeta)\phi_n(x) dx d\zeta \right) \quad (1.47)$$

$$\boldsymbol{\varphi(x,t) = \sum_{n=1}^{\infty} \phi_n(x) \frac{e^{\alpha x-(\beta+D\lambda_n{}^2)t}}{\int_0^l \phi_n(x)^2 dx} \left(\int_0^t e^{(\beta+D\lambda_n{}^2)\zeta} \int_0^l F(x,\zeta)\phi_n(x) dx d\zeta \right) + e^{-\alpha x} J(x,t)} \quad (1.48)$$

So with t = 0

$$\varphi(x,t) = Z(x,t)e^{\alpha x-\beta t} + J(x,t)e^{\alpha 1x+\beta 1t}$$
$$\varphi(x,0) = Z(x,0)e^{\alpha x} + J(x,0)e^{\alpha 1x}$$
$$\varphi(x,0)e^{-\alpha x} - J(x,0) = Z(x,0)$$
$$J(x,t) = e^{(\alpha+D\lambda^2)t0} \int_0^l (e^{-\alpha x}\emptyset(x) - J(x,t0))\phi(x)dx$$

So we are left

$$\boldsymbol{\varphi(x,t)} = \left[-\frac{\partial z}{\partial t} + D\frac{\partial^2 z}{\partial x^2} + \gamma e^{\beta t-\alpha x} \right] \boldsymbol{e^{\alpha x-\beta t}} + \left[-\frac{\partial J}{\partial t} + D\frac{\partial^2 J}{\partial x^2} \right] \boldsymbol{e^{\alpha 1x+\beta 1t}}$$

As $\boldsymbol{\beta 1 = -\beta}$

$$\boldsymbol{\varphi(x,t)} = \left(\left[-\frac{\partial z}{\partial t} + D\frac{\partial^2 z}{\partial x^2} + \gamma e^{\beta t-\alpha x} \right] + \left[-\frac{\partial J}{\partial t} + D\frac{\partial^2 J}{\partial x^2} \right] \right) \boldsymbol{e^{\alpha x-\beta t}}$$

Reflects the fundamental solution.If it were so

$$\boldsymbol{\varphi(x,t) = Z(x,t)e^{\alpha x+\beta t} + J(x,t)e^{\alpha 1x-\beta 1t}} \quad (1.49)$$

$$\boldsymbol{\varphi(x,t)} = \left(\left[-\frac{\partial z}{\partial t} + D\frac{\partial^2 z}{\partial x^2} + \gamma e^{\beta t-\alpha x} \right] + \left[-\frac{\partial J}{\partial t} + D\frac{\partial^2 J}{\partial x^2} \right] \right) \boldsymbol{e^{\alpha x+\beta t}}$$

Both are solutions.
Also if backwards as

$$\boldsymbol{\varphi(x,t) = Z(x,t)e^{\alpha x-\beta t} - J(x,t)e^{\alpha 1x+\beta 1t}} \quad (1.50)$$

We have so

$$\frac{\partial z}{\partial t} = D\frac{\partial^2 z}{\partial x^2} + fe^{\beta t-\alpha x} + \left[\left(\frac{\partial J}{\partial t} - D\frac{\partial^2 J}{\partial x^2} \right) e^{(\beta+\beta 1)t} \right]$$

With the $\alpha = \boldsymbol{\frac{u}{2D}}$ **, Y** $\boldsymbol{\beta = \frac{u^2}{4D} + \sigma \beta 1 = -\left(\frac{u^2}{4D} + \sigma\right)}$

So general transformation function Z - where J is so$\boldsymbol{\beta = \frac{u^2}{4D} + \sigma}$ **so overall we have**

$$\boldsymbol{\varphi(x,t) = Z(x,t)e^{\alpha x-\beta t} + J(x,t)e^{\alpha 1x+\beta 1t}} \quad (1.51)$$

$$\boldsymbol{\varphi(x,t) = Z(x,t)e^{\frac{u}{2D}x-\left(\frac{u^2}{4D}+\sigma\right)t} + J(x,t)e^{\frac{u}{2D}x+\left(\frac{u^2}{4D}+\sigma\right)t}} \quad (1.52)$$

So according to the diverse literature there have to reduce

$$\boldsymbol{\varphi(x,t) = Z(x,t)e^{\frac{u}{2D}x-\left(\frac{u^2}{4D}+\sigma\right)t} + J(x,t)e^{\frac{u}{2D}x}}$$

If J = 0

$$\boldsymbol{\varphi(x,t) = Z(x,t)e^{\frac{u}{2D}x-\left(\frac{u^2}{4D}+\sigma\right)t}}$$
$$\boldsymbol{\varphi(x,t) = J(x,t)e^{\frac{u}{2D}x+\left(\frac{u^2}{4D}+\sigma\right)t}}$$
$$\boldsymbol{\varphi(x,t) = Z(x,t)e^{\frac{u}{2D}x}}$$

For this particular way and as we have seen in the literature where both expressions are used
Anobservationifwe divide

$$\varphi(x,t) = 1 + \frac{J(x,t)e^{\alpha 1x+\beta 1t}}{Z(x,t)e^{\alpha x-\beta t}}$$
$$\varphi(x,t) = M(x,t)e^{\alpha 1x+\beta 1t-\alpha x+\beta t} + 1$$

A little management and have the following variables have the beta term that is more complete so if we take the absolute value of beta and maintaining the sign of the coefficient (-)
Let'ssee

$$\beta = -\beta 1 y - \beta = \beta 1$$
$$\varphi(x,t) = M(x,t)e^{-\left(\frac{u^2}{4D}+\sigma\right)t} + 1$$

And so

$$\varepsilon = \frac{u^2}{4D} + \sigma$$

If u = 0 we have the desired value of epsilon

$$\varphi(x,t) = M(x,t)e^{-\left(\frac{u^2}{4D}+\sigma\right)t} + 1 \qquad \textbf{(1.53)}$$

Another way to see that it really is to find the function to integrate the general model equation

$$\frac{\partial\varphi}{\partial t} = -u\frac{\partial\varphi}{\partial x} + \mu\frac{\partial^2\varphi}{\partial x^2} - \sigma\varphi + q(t)\delta(x-x_0)$$
$$\frac{\partial\varphi}{\partial t} - L\varphi = f$$
$$L = -u\frac{\partial}{\partial x} + \mu\frac{\partial^2}{\partial x^2} - \sigma$$

Where the solution is

$$\varphi(x,t) = Ce^{-Lt} - \frac{f}{L}$$

This transformation function is very similar to the problem number 10 of section 18.2 of the book[4] Peter V. O'Neil Advanced Engineering Mathematics, Third edition 2001, now we write in this way where L constant f / L operator

$$\varphi(x,t) = M(x,t)e^{-\varepsilon t} - 1 \qquad \textbf{(1.54)}$$

For model where only pure diffusion - reaction and forcing

$$\frac{\partial\varphi}{\partial t} = \mu\frac{\partial^2\varphi}{\partial x^2} - \sigma\varphi + q(t)\delta(x-x_0) \qquad \textbf{(1.55)}$$

$$\frac{\partial\varphi}{\partial t} = -\varepsilon M e^{-\varepsilon t} + \frac{\partial M}{\partial t}e^{-\varepsilon t}$$
$$\mu\frac{\partial^2\varphi}{\partial x^2} = \mu\frac{\partial^2 M}{\partial x^2}e^{-\varepsilon t}$$

We continue with the substitutions

$$-\varepsilon M e^{-\varepsilon t} + \frac{\partial M}{\partial t}e^{-\varepsilon t} = \mu\frac{\partial^2 M}{\partial x^2}e^{-\varepsilon t} - \sigma M e^{-\varepsilon t} + \sigma + f$$
$$\frac{\partial M}{\partial t}e^{-\varepsilon t} = \mu\frac{\partial^2 M}{\partial x^2}e^{-\varepsilon t} + \varepsilon M e^{-\varepsilon t} - \sigma M e^{-\varepsilon t} + \sigma + f$$
$$\frac{\partial M}{\partial t} = \mu\frac{\partial^2 M}{\partial x^2} + (\varepsilon - \sigma)M + (f+\sigma)e^{\varepsilon t}$$
$$\varepsilon = \sigma$$
$$\frac{\partial M}{\partial t} = \mu\frac{\partial^2 M}{\partial x^2} + (f+\sigma)e^{\varepsilon t}$$
$$\frac{\partial M}{\partial t} = \mu\frac{\partial^2 M}{\partial x^2} + e^{\varepsilon t}F$$
$$\varphi(x,0) = 0 0 < x < l$$

$$\mu\frac{\partial\varphi(0,t)}{\partial x}=0\mu\frac{\partial\varphi(l,t)}{\partial x}=0$$

$$M(x,0)=10<x<l$$
$$M(x,0)=e^{\varepsilon t}0<x<l$$

$$t\rightarrow\infty and e^{\varepsilon t}=0$$

$$M(x,0)=00<x<l$$

$$\mu\frac{\partial M(0,t)}{\partial x}e^{-\varepsilon t}=0\mu\frac{\partial M(l,t)}{\partial x}e^{-\varepsilon t}=0$$

With the boundary conditions as

$$\mu\frac{\partial\varphi(0,t)}{\partial x}+u\varphi(0,t)=0\sigma>0$$
$$\mu\frac{\partial M(0,t)}{\partial x}e^{-\varepsilon t}+u(M(0,t)e^{-\varepsilon t}-1)=0\sigma>0$$
$$\mu\frac{\partial M(0,t)}{\partial x}e^{-\varepsilon t}+uM(0,t)e^{-\varepsilon t}-u=0\sigma>0$$
$$\mu\frac{\partial M(0,t)}{\partial x}+uM(0,t)=ue^{\varepsilon t}\sigma>0$$
$$In this case ue^{\varepsilon t} with t\rightarrow\infty is iqual to 0$$
$$\mu\frac{\partial M(0,t)}{\partial x}+uM(0,t)=0\sigma>0$$

Thus the system to be solved is the following with the boundary conditions are Neumann or mixed.

$$\varphi(x,0)=00<x<l$$

$$\mu\frac{\partial\varphi(0,t)}{\partial x}=0\mu\frac{\partial\varphi(l,t)}{\partial x}=0$$

Or

$$\varphi(x,0)=00<x<l$$

$$\mu\frac{\partial\varphi(0,t)}{\partial x}+u\varphi(0,t)=0\mu\frac{\partial\varphi(l,t)}{\partial x}=0$$

With the transformation Z - J EDR number 2 in (1.55)

$$\frac{\partial M}{\partial t}=\mu\frac{\partial^2 M}{\partial x^2}+e^{\varepsilon t}F$$
$$M(x,0)=10<x<l$$

$$\mu\frac{\partial M(0,t)}{\partial x}-uM(0,t)=0$$
$$\mu\frac{\partial M(l,t)}{\partial x}=0$$

$$\frac{\partial M}{\partial t}=\mu\frac{\partial^2 M}{\partial x^2}+e^{\varepsilon t}F$$
$$M(x,0)=10<x<l$$

$$\mu\frac{\partial M(0,t)}{\partial x}e^{-\varepsilon t}=0\mu\frac{\partial M(l,t)}{\partial x}e^{-\varepsilon t}=0$$

See also the solution separation system variable

$$\frac{\partial M}{\partial t}=\mu\frac{\partial^2 M}{\partial x^2}+e^{\varepsilon t}F$$

$$M(x,0) = 10 < x < l$$

$$\mu \frac{\partial M(0,t)}{\partial x} + uM(0,t) = 0$$
$$\mu \frac{\partial M(l,t)}{\partial x} = 0$$

$$M(x,t) = \sum_{n=m}^{\infty} \cos(x\lambda) \left[\int_0^t d_k Q(\tau) e^{-(\lambda^2\mu+\sigma)(t-\tau)} d\tau \right] e^{\varepsilon t} + 1$$

With the Fourier coefficient given as

$$d_k = \frac{cos(x0\lambda)}{\left[\frac{1}{2}\left(1 + \frac{sen(2\lambda)}{2\lambda}\right)\right]}$$

To converge the coefficient should apply his series of Fejer, this solution due to the initial condition at t = 0 and border M

$$\varphi(x,t) = \left[\sum_{n=m}^{\infty} \cos(x\lambda) \left[\int_0^t d_k Q(\tau) e^{-(\lambda^2\mu+\sigma)(t-\tau)} d\tau \right] e^{\varepsilon t} + 1 \right] e^{-\varepsilon t} - 1 \tag{1.56}$$

When t = 0

$$\varphi(x,t) = \left[\sum_{n=m}^{\infty} \cos(x\lambda) \left[\int_0^t d_k Q(\tau) e^{-(\lambda^2\mu+\sigma)(t-\tau)} d\tau \right] + e^{-\varepsilon t} \right] - 1 = 0$$

Is fulfilled

Zenteno – Julia Functions

Table.1

$\varphi(x,t) = Z(x,t)e^{\alpha x-\beta t} + J(x,t)e^{\alpha 1x+\beta 1t}$
$\varphi(x,t) = Z(x,t)e^{\alpha x-\beta t} - J(x,t)e^{\alpha 1x+\beta 1t}$
$\varphi(x,t) = M(x,t)e^{-\varepsilon t} - C$

Conclusions

According to various sources to transform or reduce of advection diffusion reaction equation to a simpler way to solve it, the exponential function is very important, we can also see that by integrating the equation with its various terms the same integral factor is within of the same solution of the equation, we can also see that having a different initial condition of zero and with the boundary conditions inhomogeneous solution only output increased or decreased depending on the diffusion coefficients, advection or reaction, also a little above the time axis or more below this, it is very susceptible to these conditions.

The options proposed now Zenteno – Julia functionsare just a supplement that isalready known, and that in some applications can see reduced the equation with both separately and now found that together give more meaning to the solution and get the same, keeping the coefficient alpha as well, this solution has a point in the delta function, therefore it affects quite away from the point of origin.
The third function is already known to the form presented only that we can now add a constant in this case was the unit or simply a factor C, giving the same results and where possibly obtained.

$$\varphi(x,t) = M(x,t)e^{-\varepsilon t} - C$$

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