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Functions for Reduction of Parabolic Hyperbolic Equations Zenteno – Julia Functions and some applications

M. Sc. Zenteno Jiménez José Roberto
Geophysical Engineering, National Polytechnic Institute, México City,
ESIA-Ticóman Unit Mayor Gustavo A. Madero

Abstract: The following topic is about the case or cases of reduction of partial differential hyperbolic parabolic equations of linear second order, in the traditional literature there are functions or substitution methods to reduce certain expressions but in many of these cases it is unknown because use a certain function and under what sign some variables remain within the transformation function, here there is only a set of expressions that reduce the equations with a general transformation function, the transformation functions are already known in the literature of differential equations, Emphasis is made only on why these expressions can be used for use.
Keywords: Parabolic - Hyperbolic Partial Differential Equations and Wronskian

Introduction

Linear Parabolic Differential Equations and some Hyperbolic have enough methods to reduce the expression to a simpler solution and in the literature we find several methods or functions to reduce, in many of them the same function intervenes except for the signs of the exponents if necessary for its implementation, then the subject to be analyzed is a function or set of functions that reduce the Parabolic or Hyperbolic PDE, having in each case the two functions with the different sign each.

The next proposed transformation is as follows we can see that the solution for these equations of this type tipo **(x, t) = X (x, t) * Y (x, t)** so let's propose the following form**ϕ (x, t) = Z (x, t) * X (x, t) + J (x, t) * Y (x, t)** we know that one of them is exponentially being X or Y so to find the solution having the part non-homogeneous and also a way to obtain all the transformation functions used and for this case, where the only thing that varies is the sign of the coefficients in the exponent.

Transformation Function

Whether it is a function that we want to use to reduce homogeneous Linear Parabolic PDE or not to a simpler way, we have the following

$$\boldsymbol{\varphi(x,y) = Z(x,y) * J(x,y)} \qquad \textbf{(1)}$$

Where we do not know any of the two functions, so we substitute in the EDP and you have the following

$$ao\frac{d^2\varphi}{dx^2} + a1\frac{d\varphi}{dx} + a2\varphi = 0$$

With the coefficients a = 1 with

$$\varphi(x) = Z(x)$$

And it gives us an expression like the one below

$$Z(x) = Ce^{-x}$$

But a complex solution, which that part does not matter to us at the moment
If we now substitute the following

$$\varphi(x,y) = Z(x,y)$$

The solution is similar if one of the variables is set as constant in this case and

$$Z(x,y) = Ce^{-x}$$

Now if we substitute in the homogeneous linear Parabolic PDE and with a = 1 with the same substitution

$$\varphi(x,t) = Z(x,t)$$

$$\frac{\partial\varphi(x,t)}{\partial t} = a0\frac{\partial^2\varphi(x,t)}{\partial x^2} + a1\frac{\partial\varphi(x,t)}{\partial x} + a2\varphi(x,t)$$

The solution will be something like, depending on whether we match 1 or -1

$$Z(x,t) = Ce^{-x}$$
$$Z(x,t) = Ce^{-t}$$
$$Z(x,t) = Ce^{t}$$

Then the reduction function can be obtained as follows

$$Z(x,t) = Ce^{-(x+t)}$$

And the other like

$$Z(x,t) = Ce^{t-x}$$

According to several expressions of reduction for this type of equations we see that an important factor is the exponential that accompanies it with a + or sign - so we can think of a function of this style.
Here are some known examples

Table 1

Linear PDE Parabolic	Known Transformation Function
$\frac{\partial\varphi(x,t)}{\partial t} = \frac{\partial^2\varphi(x,t)}{\partial x^2} + \varphi(x,t) + A$	$\varphi(x,t) = Z(x,t)e^{\alpha x}$ $\varphi(x,t) = Z(x,t)e^{at}$
$\frac{\partial\varphi(x,t)}{\partial t} = \frac{\partial^2\varphi(x,t)}{\partial x^2} + \frac{\partial\varphi(x,t)}{\partial x} + \varphi(x,t)$	$\varphi(x,t) = Z(x,t)e^{\alpha x-\beta t}$ $\varphi(x,t) = Z(x,t)e^{\alpha x+\beta t}$
$\frac{\partial\varphi(x,t)}{\partial t} = \frac{\partial^2\varphi(x,t)}{\partial x^2} + \frac{\partial\varphi(x,t)}{\partial x} + \varphi(x,t) + A$	$\varphi(x,t) = Z(x,t)e^{\alpha x-\beta t} + J(x,t)e^{\alpha x}$

Then we can propose the following Function

$$\boldsymbol{\varphi(x,y) = Z(x,y) * J(x,y) + Z(x,y) * J(x,y)} \quad \textbf{(2)}$$

The combination of both linearly added, where one we know is the exponential, the function can be in the following terms

$$\boldsymbol{\varphi(x,y) = Z(x,y) * e^{\alpha x-\beta t} + J(x,y) * e^{\alpha x+\beta t}} \quad \textbf{(3)}$$

Where we don´t know the other function but it will serve to reduce the resulting equation, but we can also write a game of themselves as

Zenteno – Julia Functions
Table 2 Transformation Functions

$\boldsymbol{\varphi(x,y) = Z(x,y) * e^{\alpha x-\beta t} + J(x,y) * e^{\alpha x+\beta t}}$
$\boldsymbol{\varphi(x,y) = Z(x,y) * e^{\alpha x-\beta t} - J(x,y) * e^{\alpha x+\beta t}}$

If we calculate its determinant, it gives us the following

$$\left[Z(x,y) * e^{\alpha x-\beta t} J(x,y) * e^{\alpha x+\beta t} Z(x,y) * e^{\alpha x-\beta t} - J(x,y) * e^{\alpha x+\beta t}\right]$$
$$= -Z(x,y) * e^{\alpha x-\beta t} * J(x,y) * e^{\alpha x+\beta t} - J(x,y) * e^{\alpha x+\beta t} Z(x,y) * e^{\alpha x-\beta t}$$
$$= -Z(x,y) * J(x,y) * e^{2\alpha x} - J(x,y) * Z(x,y) * e^{2\alpha x}$$
$$= -2Z(x,y) * J(x,y) * e^{2\alpha x}$$

Functions are linearly independent if and only if the determinant of the matrix formed by vectors as columns is non-zero.
Now let's look at the Reduced Equation by applying this proposal.

Table 3

Parabolic PDE	Transformation Function
$\frac{\partial\varphi(x,t)}{\partial t} + \frac{\partial\varphi(x,t)}{\partial x} = \frac{\partial^2\varphi(x,t)}{\partial x^2}$	The function remains as: $\varphi(x,y) = Z(x,y) * e^{\frac{1}{2}x+\frac{3}{4}t} + J(x,y) * e^{\frac{1}{2}x-\frac{3}{4}t}$

The Reduced Equation is:

$$\frac{\partial Z(x,t)}{\partial t} = \frac{\partial^2 Z(x,t)}{\partial x^2} + \left[-\frac{\partial J(x,t)}{\partial t} + \frac{\partial^2 J(x,t)}{\partial x^2}\right]$$

$$\frac{\partial\varphi(x,t)}{\partial t}=\frac{\partial^2\varphi(x,t)}{\partial x^2}-\frac{\partial\varphi(x,t)}{\partial x}$$

The Reduced Equation is:

$$\frac{\partial Z(x,t)}{\partial t}=\frac{\partial^2 Z(x,t)}{\partial x^2}+\left[-\frac{\partial J(x,t)}{\partial t}+\frac{\partial^2 J(x,t)}{\partial x^2}\right]$$

The function remains as:

$$\varphi(x,y)=Z(x,y)*e^{\frac{1}{2}x-\frac{3}{4}t}+J(x,y)*e^{\frac{1}{2}x-\frac{3}{4}t}$$

$$\frac{\partial\varphi(x,t)}{\partial t}=\frac{\partial^2\varphi(x,t)}{\partial x^2}-\varphi(x,t)$$

The Reduced Equation is:

$$\frac{\partial Z(x,t)}{\partial t}=\frac{\partial^2 Z(x,t)}{\partial x^2}+\left[-\frac{\partial J(x,t)}{\partial t}+\frac{\partial^2 J(x,t)}{\partial x^2}\right]$$

The function remains as:

$$\varphi(x,y)=Z(x,y)*e^{-t}+J(x,y)*e^{-t}$$

With $+\varphi(x,t)$

$$\varphi(x,y)=Z(x,y)*e^{t}+J(x,y)*e^{t}$$

$$\frac{\partial\varphi(x,t)}{\partial t}=\frac{\partial^2\varphi(x,t)}{\partial x^2}+\varphi(x,t)+A$$

The Reduced Equation is:

$$\frac{\partial Z(x,t)}{\partial t}=\frac{\partial^2 Z(x,t)}{\partial x^2}+\left[-\frac{\partial J(x,t)}{\partial t}+\frac{\partial^2 J(x,t)}{\partial x^2}\right]e^{2t}+Ae^{-t}$$

The function remains as:

$$\varphi(x,y)=Z(x,y)*e^{t}+J(x,y)*e^{t}$$

Con $-\varphi(x,t)$

$$\varphi(x,y)=Z(x,y)*e^{-t}+J(x,y)*e^{-t}$$

We can use only Z or J

$$\frac{\partial Z(x,t)}{\partial t}=\frac{\partial^2 Z(x,t)}{\partial x^2}+Ae^{-t}$$

$$\frac{\partial\varphi(x,t)}{\partial t}=\frac{\partial^2\varphi(x,t)}{\partial x^2}+\frac{\partial\varphi(x,t)}{\partial x}+\varphi(x,t)$$

The Reduced Equation is:

$$\frac{\partial Z(x,t)}{\partial t}=\frac{\partial^2 Z(x,t)}{\partial x^2}+\left[-\frac{\partial J(x,t)}{\partial t}+\frac{\partial^2 J(x,t)}{\partial x^2}\right]e^{\frac{3}{2}t}$$

The function remains as:

$$\varphi(x,y)=Z(x,y)*e^{-\frac{1}{2}x+\frac{3}{4}t}+J(x,y)*e^{-\frac{1}{2}x+\frac{3}{4}t}$$

We can use only Z or J

$$\frac{\partial Z(x,t)}{\partial t}=\frac{\partial^2 Z(x,t)}{\partial x^2}+Ae^{\frac{1}{2}x-\frac{3}{4}t}$$

With $-\frac{\partial\varphi(x,t)}{\partial x}-\varphi(x,t)$

$$\varphi(x,y) = Z(x,y) * e^{\frac{1}{2}x-\frac{5}{4}t} + J(x,y) * e^{\frac{1}{2}x-\frac{7}{4}t}$$

$$\frac{\partial \varphi(x,t)}{\partial t} = \frac{\partial^2 \varphi(x,t)}{\partial x^2} + \frac{\partial \varphi(x,t)}{\partial x} + \varphi(x,t) + A$$

The Reduced Equation is:

$$\frac{\partial Z(x,t)}{\partial t} = \frac{\partial^2 Z(x,t)}{\partial x^2} + \left[-\frac{\partial J(x,t)}{\partial t} + \frac{\partial^2 J(x,t)}{\partial x^2}\right] e^{\frac{3}{2}t} + Ae^{\frac{1}{2}x-\frac{5}{4}t}$$

The function remains as:

$$\varphi(x,y) = Z(x,y) * e^{-\frac{1}{2}x+\frac{3}{4}t} + J(x,y) * e^{-\frac{1}{2}x+\frac{3}{4}t}$$

We can use only Z or J

$$\frac{\partial Z(x,t)}{\partial t} = \frac{\partial^2 Z(x,t)}{\partial x^2} + Ae^{\frac{1}{2}x-\frac{5}{4}t}$$

With $-\frac{\partial \varphi(x,t)}{\partial x} - \varphi(x,t) - A$

$$\varphi(x,y) = Z(x,y) * e^{\frac{1}{2}x-\frac{5}{4}t} + J(x,y) * e^{\frac{1}{2}x-\frac{7}{4}t}$$

We must realize something by reducing these expressions that Function Z and J are the same, but because the coefficients of the terms of the Equations in Partial Derivatives such as Advection, Diffusion or Reaction are equal to 1 and positive, and we can see what sign It is the one that dominates, as in the following case and we see how the signs change in the exponential.
Now let's apply the same reduction function but for Hyperbolic Partial Differential Equations.

Table 4.

Hyperbólic Equation

Non-homogeneous Hyperbolic Diffusion Equation

$$\frac{\partial^2 \varphi(x,t)}{\partial t^2} + \frac{\partial \varphi(x,t)}{\partial t} - \frac{\partial^2 \varphi(x,t)}{\partial x^2} = A$$

Equation of Non-Homogeneous Wave Reduction without J, the Klein Gordon Equation

$$\frac{\partial^2 Z(x,t)}{\partial t^2} = \frac{\partial^2 Z(x,t)}{\partial x^2} + a_T Z(x,t) + Ae^{\pm\frac{1}{2}t}$$

The function remains as:

$$\varphi(x,t) = Z(x,t)e^{-\frac{1}{2}t} + J(x,t)e^{-\frac{1}{2}t}$$

With

$$\frac{\partial^2 \varphi(x,t)}{\partial t^2} - \frac{\partial \varphi(x,t)}{\partial t} - \frac{\partial^2 \varphi(x,t)}{\partial x^2} = A$$

$$\varphi(x,t) = Z(x,t)e^{\frac{1}{2}t} + J(x,t)e^{\frac{1}{2}t}$$

Non-homogeneous Hyperbolic Diffusion Equation

$$\frac{\partial^2 \varphi(x,t)}{\partial t^2} + \frac{\partial \varphi(x,t)}{\partial t} - \frac{\partial^2 \varphi(x,t)}{\partial x^2} + \varphi(x,t) = 0$$

The Reduction Equation is the Klein Gordon Equation

$$\frac{\partial^2 Z(x,t)}{\partial t^2} = \frac{\partial^2 Z(x,t)}{\partial x^2} + a_T Z(x,t)$$

The function remains as:

$$\varphi(x,t) = Z(x,t)e^{-\frac{1}{2}t} + J(x,t)e^{-\frac{1}{2}t}$$

With

$$\frac{\partial^2 \varphi(x,t)}{\partial t^2} + \frac{\partial \varphi(x,t)}{\partial t} - \frac{\partial^2 \varphi(x,t)}{\partial x^2} - \varphi(x,t) = 0$$

$$\varphi(x,t) = Z(x,t)e^{-\frac{1}{2}t} + J(x,t)e^{-\frac{1}{2}t}$$

Table 5.

Hyperbolic Diffusion Equation

$$\frac{\partial^2\varphi(x,t)}{\partial t^2}+\frac{\partial\varphi(x,t)}{\partial t}-\frac{\partial^2\varphi(x,t)}{\partial x^2}+\frac{\partial\varphi(x,t)}{\partial x}=0$$

The Reduction Equation is the Homogeneous Wave Equation without J

$$\frac{\partial^2 Z(x,t)}{\partial t^2}=\frac{\partial^2 Z(x,t)}{\partial x^2}$$

The function remains as:

$$\varphi(x,t)=Z(x,t)e^{\frac{1}{2}(x-t)}+J(x,t)e^{\frac{1}{2}(x+t)}$$

With

$$\frac{\partial^2\varphi(x,t)}{\partial t^2}-\frac{\partial\varphi(x,t)}{\partial t}-\frac{\partial^2\varphi(x,t)}{\partial x^2}+\frac{\partial\varphi(x,t)}{\partial x}=0$$

$$\varphi(x,t)=Z(x,t)e^{\frac{1}{2}(x+t)}+J(x,t)e^{\frac{1}{2}(x+t)}$$

With

$$\frac{\partial^2\varphi(x,t)}{\partial t^2}-\frac{\partial\varphi(x,t)}{\partial t}-\frac{\partial^2\varphi(x,t)}{\partial x^2}-\frac{\partial\varphi(x,t)}{\partial x}=0$$

$$\varphi(x,t)=Z(x,t)e^{-\frac{1}{2}x+\frac{1}{2}t}+J(x,t)e^{-\frac{1}{2}x+\frac{1}{2}t}$$

Hyperbolic Equation of Advection Diffusion Non-homogeneous reaction

$$\frac{\partial^2\varphi(x,t)}{\partial t^2}+\frac{\partial\varphi(x,t)}{\partial t}-\frac{\partial^2\varphi(x,t)}{\partial x^2}+\frac{\partial\varphi(x,t)}{\partial x}=-\varphi(x,t)+A$$

The Reduction Equation is the non-homogeneous Klein Gordon Equation

$$\frac{\partial^2 Z(x,t)}{\partial t^2}=\frac{\partial^2 Z(x,t)}{\partial x^2}+a_t Z(x,t)+Ae^{-\alpha x+\beta t}$$

The function remains as:

$$\varphi(x,t)=Z(x,t)e^{\frac{1}{2}(x-t)}+J(x,t)e^{\frac{1}{2}(x+t)}$$

With

$$\frac{\partial^2\varphi(x,t)}{\partial t^2}-\frac{\partial\varphi(x,t)}{\partial t}-\frac{\partial^2\varphi(x,t)}{\partial x^2}+\frac{\partial\varphi(x,t)}{\partial x}=-\varphi(x,t)+A$$

$$\varphi(x,t)=Z(x,t)e^{\frac{1}{2}x+\frac{1}{2}t}+J(x,t)e^{\frac{1}{2}x+\frac{1}{2}t}$$

With

$$\frac{\partial^2\varphi(x,t)}{\partial t^2}-\frac{\partial\varphi(x,t)}{\partial t}-\frac{\partial^2\varphi(x,t)}{\partial x^2}-\frac{\partial\varphi(x,t)}{\partial x}=+\varphi(x,t)+A$$

$$\varphi(x,t)=Z(x,t)e^{-\frac{1}{2}x+\frac{1}{2}t}+J(x,t)e^{-\frac{1}{2}x+\frac{1}{2}t}$$

Homogeneous Hyperbolic Diffusion Equation

$$\frac{\partial^2\varphi(x,t)}{\partial t^2}+\frac{\partial\varphi(x,t)}{\partial t}-\frac{\partial^2\varphi(x,t)}{\partial x^2}+\varphi(x,t)=0$$

The Reduction Equation is the Klein Gordon Equation

$$\frac{\partial^2 Z(x,t)}{\partial t^2}=\frac{\partial^2 Z(x,t)}{\partial x^2}+a_t Z(x,t)$$

The function remains as:

$$\varphi(x,t)=Z(x,t)e^{-\frac{1}{2}t}+J(x,t)e^{-\frac{1}{2}t}$$

With

$$\frac{\partial^2\varphi(x,t)}{\partial t^2}-\frac{\partial\varphi(x,t)}{\partial t}-\frac{\partial^2\varphi(x,t)}{\partial x^2}-\varphi(x,t)=0$$

$$\varphi(x,t) = Z(x,t)e^{\frac{1}{2}t} + J(x,t)e^{\frac{1}{2}t}$$

Transformation Functions

Table 6.

Parabolic PDE	Hyperbolic PDE
$\boldsymbol{\varphi(x,y) = Z(x,y)e^{\frac{1}{2}x+\frac{3}{4}t} + J(x,y)e^{\frac{1}{2}x-\frac{3}{4}t}}$ $\boldsymbol{\varphi(x,y) = Z(x,y)e^{\frac{1}{2}x-\frac{3}{4}t} + J(x,y)e^{\frac{1}{2}x-\frac{3}{4}t}}$	$\varphi(x,t) = Z(x,t)e^{-\frac{1}{2}t} + J(x,t)e^{-\frac{1}{2}t}$ $\varphi(x,t) = Z(x,t)e^{\frac{1}{2}t} + J(x,t)e^{\frac{1}{2}t}$
$\boldsymbol{\varphi(x,y) = Z(x,y)e^{-t} + J(x,y)e^{-t}}$ $\boldsymbol{\varphi(x,y) = Z(x,y)e^{t} + J(x,y)e^{t}}$	$\varphi(x,t) = Z(x,t)e^{\frac{1}{2}(x-t)} + J(x,t)e^{\frac{1}{2}(x+t)}$ $\varphi(x,t) = Z(x,t)e^{\frac{1}{2}(x+t)} + J(x,t)e^{\frac{1}{2}(x+t)}$ $\varphi(x,t) = Z(x,t)e^{-\frac{1}{2}x+\frac{1}{2}t} + J(x,t)e^{-\frac{1}{2}x+\frac{1}{2}t}$
$\boldsymbol{\varphi(x,y) = Z(x,y)e^{-\frac{1}{2}x+\frac{3}{4}t} + J(x,y)e^{-\frac{1}{2}x+\frac{3}{4}t}}$ $\boldsymbol{\varphi(x,y) = Z(x,y)e^{\frac{1}{2}x-\frac{5}{4}t} + J(x,y)e^{\frac{1}{2}x-\frac{7}{4}t}}$	$\varphi(x,t) = Z(x,t)e^{\frac{1}{2}(x-t)} + J(x,t)e^{\frac{1}{2}(x+t)}$ $\varphi(x,t) = Z(x,t)e^{\frac{1}{2}x+\frac{1}{2}t} + J(x,t)e^{\frac{1}{2}x+\frac{1}{2}t}$ $\varphi(x,t) = Z(x,t)e^{-\frac{1}{2}x+\frac{1}{2}t} + J(x,t)e^{-\frac{1}{2}x+\frac{1}{2}t}$
	$\varphi(x,t) = Z(x,t)e^{-\frac{1}{2}t} + J(x,t)e^{-\frac{1}{2}t}$ $\varphi(x,t) = Z(x,t)e^{\frac{1}{2}t} + J(x,t)e^{\frac{1}{2}t}$

Let's take a look at the properties of these functions with a well-known theorem within the ODE. The Wronskian, which we see if the solutions functions are dependent or linearly independent.

The Wronskians are functions so named in honor of the Polish physicist, philosopher and mathematician Josef Hoene-Wronski (1778-1853). Fundamental in the study of differential equation systems. These systems arise in problems that are mainly related to the dependent variables which are a function of the independent variable itself. They are useful to determine if two functions are linearly independent and thus create a solution set that at the same time respects the theory of differential equations, these reduction functions with constant coefficient equal to 1 except with changes in the sign are solutions of The same equation will then see how dependent or independent they are:

Table 7.

Reduction Function	Wronskiano.
$\boldsymbol{\varphi(x,t) = Z(x,t)e^{\frac{1}{2}x+\frac{3}{4}t} + J(x,t)e^{\frac{1}{2}x-\frac{3}{4}t}}$	With t cte $\left[Z(x,t)e^{\frac{1}{2}x+\frac{3}{4}t}J(x,t)e^{\frac{1}{2}x-\frac{3}{4}t}Z(x,t)_x e^{\frac{1}{2}x+\frac{3}{4}t} + \frac{1}{2}Z(x,t)e^{\frac{1}{2}x+\frac{3}{4}t}J(x,t)_x e^{\frac{1}{2}x-\frac{3}{4}t} + \frac{1}{2}J(x,t)e^{\frac{1}{2}x-\frac{3}{4}t}\right]$ $\neq 0$
Regarding t and now x is cte.	$\left[Z(x,t)e^{\frac{1}{2}x+\frac{3}{4}t}J(x,t)e^{\frac{1}{2}x-\frac{3}{4}t}Z(x,t)_t e^{\frac{1}{2}x+\frac{3}{4}t} + \frac{3}{4}Z(x,t)e^{\frac{1}{2}x+\frac{3}{4}t}J(x,t)_t e^{\frac{1}{2}x-\frac{3}{4}t} - \frac{3}{4}J(x,t)e^{\frac{1}{2}x-\frac{3}{4}t}\right]$ $\neq 0$
$\boldsymbol{\varphi(x,t) = Z(x,t)e^{-t} + J(x,t)e^{-t}}$	With t cte $[Z(x,t)e^{-t}J(x,t)e^{-t} - Z(x,t)e^{-t} + Z(x,t)_t e^{-t} - J(x,t)e^{-t} + J(x,t)_t e^{-t}] = 0$ What time dependence indicates

Regarding t and now x is cte. $[Z(x,t)e^{-t}J(x,t)e^{-t}Z(x,t)_x e^{-t}J(x,t)_x e^{-t}] \neq 0$

With these tests of some of them we can verify that depending on the sign found in the exponents of the Exponential we see how dependent the solutions are with respect to space or time.
Now the functions are given as follows

$$\varphi(x,y) = Z(x,y) * e^{\alpha x-\beta t} + J(x,y) * e^{\alpha x+\beta t}$$
$$\varphi(x,y) = Z(x,y) * e^{\alpha x-\beta t} - J(x,y) * e^{\alpha x+\beta t}$$

If we add them, it gives us the following

$$2\varphi(x,y) = 2Z(x,y) * e^{\alpha x-\beta t}$$
$$\varphi(x,y) = Z(x,y) * e^{\alpha x-\beta t}$$

Or changing the sign

$$\varphi(x,y) = Z(x,y) * e^{\alpha x-\beta t} + J(x,y) * e^{\alpha x+\beta t}$$
$$\varphi(x,y) = -Z(x,y) * e^{\alpha x-\beta t} + J(x,y) * e^{\alpha x+\beta t}$$

If we add them, it gives us the following

$$2\varphi(x,y) = 2J(x,y) * e^{\alpha x+\beta t}$$
$$\varphi(x,y) = J(x,y) * e^{\alpha x+\beta t}$$

We obtain the Known Transformation Functions, depending on x, t or both functions if it is unknown.

Applications

Table 8.

Applied Equation	
Advection equation Diffusion Non-homogeneous reaction $$\frac{\partial\varphi(x,t)}{\partial t} = \frac{\partial^2\varphi(x,t)}{\partial x^2} - \frac{\partial\varphi(x,t)}{\partial x} - \varphi(x,t) + A$$	Case where for reduction can be used $$\varphi(x,t) = Z(x,t) * e^{\alpha x-\beta t} + J(x,t) * e^{\alpha x}$$
Reduced Equation	Article of the implementation of the solution
Non-Homogeneous Diffusion Equation $$\frac{\partial Z(x,t)}{\partial t} = \frac{\partial^2 Z(x,t)}{\partial x^2} + [-J_t + J_{xx}]e^{2\beta t} + Ae^{-\alpha x+\beta t}$$	https://www.researchgate.net/publication/325561086_Una_solucion_analitica_para_la_ecuacion_de_difusion_advec cion_reaccion_por_medio_de_la_serie_de_Fourier M. en C. Zenteno Jimenez Jose Roberto (Revista Matemática Latinoamérica)
$$\frac{\partial\varphi}{\partial t} + u\frac{\partial\varphi}{\partial x} - \mu\frac{\partial^2\varphi}{\partial x^2} + \sigma\varphi = q(t)\delta(x-x_0)$$ $$\varphi(x,0) = \varphi_0 \quad 0 < x < l$$ $$\mu\frac{\partial\varphi(l,t)}{\partial x} - u\varphi(l,t) = C1 \qquad t > 0$$ $$\mu\frac{\partial\varphi(0,t)}{\partial x} - u\varphi(0,t) = C2 \qquad \sigma > 0$$ **Case where for Reduction can be used** $$\varphi(x,t) = w(x,t)e^{rx-st} + e^{rx}H(x,t)$$	General Analytical Solution Unidimensional Advection-Diffusion Reaction Equation Inhomogeneous on a Bounded Domain with an Application in Dispersion of Pollutants International Journal of Latest Research in Engineering and Technology (IJLRET) ISSN: 2454-5031 www.ijlret.com || Volume 05 - Issue 02 || February 2019 || PP. 17-33 http://www.ijlret.com/Papers/Vol-05-issue-02/2.B2019005.pdf

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} + e^{st}F(x,t)$$

$$with\, F(x,t) = \gamma e^{-rx} - \frac{\partial H}{\partial t} + D\frac{\partial^2 H}{\partial x^2}$$

Equation of the Linear Model Black - Scholes

$$\frac{\partial \varphi(s,t)}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \varphi(s,t)}{\partial s^2} + rs\frac{\partial \varphi(s,t)}{\partial s} - r\varphi(s,t) = 0$$

Case where for reduction can be used

$$\varphi(s,t) = Z(s,y) * e^{\alpha s - \beta t} + J(s,y) * e^{\alpha s + \beta t}$$

Reduced Equation

$$\frac{\partial Z(s,t)}{\partial t} = -\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 Z(s,t)}{\partial s^2} + \left[-J_t - \frac{1}{2}\sigma^2 s^2 J_{ss}\right]$$

Solution of the Black-Scholes Equation to calculate the price of an option with general payment by means of elementary techniques: Substitution and Fourier Transformation.

http://esdocs.com/doc/20987/soluci%C3%B3n-de-la-ecuaci%C3%B3n-de-black-scholes-para-calcular-el...

$$\varphi(s,t) = Z(s,t)e^{-\frac{r}{\sigma^2}+\left(\frac{1r^2}{2\sigma^2}+r\right)t} + J(s,t)e^{-\frac{r}{\sigma^2}+\left(\frac{1r^2}{2\sigma^2}+r\right)t}$$

Application for Burgers Nonlinear Equation

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2} + \varphi\frac{\partial \varphi}{\partial x}$$

Applying the product of the two functions $\varphi(x,y) = Z(x,y)J(x,y)$

$$Z_t J + Z J_t = (Z_{xx}J + 2Z_x J_x + Z J_{xx}) + ZJ(Z_x J + Z J_x)$$

Accommodating terms

$$J_t = J_{xx} + \left(\frac{2Z_x}{Z} + ZJ\right)J_x + \left(\frac{Z_{xx}}{Z} - \frac{Z_t}{Z} + Z_x\right)J$$

Now you have the following Differential Equations

$$\frac{2}{Z}\frac{\partial Z}{\partial x} + ZJ = 0 \quad \rightarrow J = -\frac{2}{Z^2}\frac{\partial Z}{\partial x}$$

$$\left(\frac{Z_{xx}}{Z} - \frac{Z_t}{Z} + Z_x\right)J = 0 \quad \rightarrow J = 0 \quad con\, \frac{2}{Z^2}\frac{\partial Z}{\partial x} = 0$$

Finally you have

$$\varphi(x,y) = Z\left(\frac{2}{Z^2}\frac{\partial Z}{\partial x}\right) = \frac{2}{Z}\frac{\partial Z}{\partial x}$$

Which is the Transformation Function through the Hopf – Cole Transformation, where Z is a solution of the Equation $J_t = J_{xx}$ reduced, see [12]

Conclusions

According to what was seen above with respect to the reduction functions, only one way was given to generalize said function and to be able to use it according to the relevant Partial Differential Equation, these resulting functions are already known as in some references and articles mentioned , and we make the observation of how to use it according to the signs of the coefficients in each Equation, we can also observe the part of the function J and how it also takes a role in the reduction function.

Discussion

Let's do the following Observation

We derive the transformation function with respect to x and t the part Z

$$\frac{\partial \varphi(x,t)}{\partial x} = aZ(x,t)e^{ax-st} + \frac{\partial Z(x,t)}{\partial x}e^{ax-st}$$
$$\frac{\partial \varphi(x,t)}{\partial t} = -sZ(x,t)e^{ax-st} + \frac{\partial Z(x,t)}{\partial t}e^{ax-st}$$

Substituting part Z

$$\frac{\partial \varphi(x,t)}{\partial x} = a\varphi(x,t) + \frac{\partial Z(x,t)}{\partial x}e^{ax-st}$$
$$\frac{\partial \varphi(x,t)}{\partial t} = -s\varphi(x,t) + \frac{\partial Z(x,t)}{\partial t}e^{ax-st}$$

Adding both

$$\frac{\partial \varphi(x,t)}{\partial x} + \frac{\partial \varphi(x,t)}{\partial t} = (a-s)\varphi(x,t) + \left[\frac{\partial Z(x,t)}{\partial x} + \frac{\partial Z(x,t)}{\partial t}\right]e^{ax-st}$$

So having the part of J

$$\frac{\partial \varphi(x,t)}{\partial x} + \frac{\partial \varphi(x,t)}{\partial t} = (a+s)\varphi(x,t) + \left[\frac{\partial J(x,t)}{\partial x} + \frac{\partial J(x,t)}{\partial t}\right]e^{ax+st}$$

Finally we get the following, the function responds to the solution according to the corresponding Equation

$$\frac{\partial \varphi(x,t)}{\partial x} + \frac{\partial \varphi(x,t)}{\partial t} = a\varphi(x,t) + \frac{1}{2}\left[\frac{\partial Z(x,t)}{\partial x} + \frac{\partial Z(x,t)}{\partial t}\right]e^{ax-st} + \frac{1}{2}\left[\frac{\partial J(x,t)}{\partial x} + \frac{\partial J(x,t)}{\partial t}\right]e^{ax+st}$$

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