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Fractional Taylor Series Based on Jumarie Type of Modified Riemann-Liouville Derivatives

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Abstract: In this paper, based on the Jumarie type of modified Riemann-Liouville (R-L) fractional derivatives, we use a new multiplication, fractional Taylor series method and chain rule for fractional derivatives to find the fractional Taylor series expansions of some fractional functions.These results we obtained are the generalizations of Taylor series expansions of several classical functions.

Keywords: Jumarie type of R-L derivatives,new multiplication,fractional Taylor series, chain rule

I. INTRODUCTION

The calculus founded by Newton and Leibniz is a very important scientific achievement in the history of mathematics. Fractional calculus was first proposed by the famous mathematician Hospital in 1695. A question is about what is $\frac{d^{1/2}x}{dx^{1/2}}$? After 124 years, Lacroix gave the right answer to this question for the first time that $\frac{d^{1/2}x}{dx^{1/2}}=\frac{2}{\sqrt{\pi}}x^{1/2}$. However, for a long time, due to the lack of practical application, fractional calculus has not been widely used. With the development of science and technology, especially since the 20th century, the theory and application of fractional calculus began to be widely concerned. Professor Mandelbrot of Yale University points out that there are many fractal dimensions in nature and science. Fractional calculus has become a powerful tool to study fractional differential equations and fractional functions, and has been widely used in the research of quantum mechanics, electrical engineering, fluid science,viscoelasticity, control theory, dynamics, finance, and so on([1-12]).On the other hand, the applications of fractional calculus to fractional differential equations can be found in [24-31].

Taylor series method is a useful tool to approximate solutions of the ordinary differential equations ([13-15]) or solutionsof thepartial differential equations ([16-17]). One advantage of the analyticmethods is that the accuracy of solution can be evaluated directly. Thus the approximatesolution can be replaced into the equation and the initial or boundary conditions.On the other hand, Jumarie([21]) have derived an expression for Taylor series of fractional order for nondifferentiable functions. By using fractional Taylor series method to approximate solutions for fractional differential equations, based on the corresponding Taylor's formula can be found in ([18-20]).Our purpose in this paper is to improve the definition of fractional Taylor series provided by Jumarie and find the fractional Taylor series of some fractional functions.Moreover, Mittag-Leffler function plays an important role in this article, which is similar to the exponential function in classical calculus.

II. PRELIMINARIES AND METHODS

At First, the fractional calculus used in this article is introduced below.

Definition 2.1: Suppose thatα is a real number and$p$ is a positive integer, then the modified Riemann-Liouville fractional derivatives of Jumarie type ([21]) is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\begin{cases}\frac{1}{\Gamma(-\alpha)}\int_{x_0}^{x}(x-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha<0\\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}(x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0\le\alpha<1\\ \frac{d^p}{dx^p}\left({}_{x_0}D_x^{\alpha-p}\right)[f(x)], & \text{if } p\le\alpha<p+1\end{cases} \quad (1)$$

where$\Gamma(u)=\int_0^{\infty}t^{u-1}e^{-t}dt$ is the gamma function defined on $u>0$. If $\left({}_{x_0}D_x^{\alpha}\right)^n[f(x)]=\left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$ exists, then $f(x)$ is called n-th order α-fractional differentiable function, and $\left({}_{x_0}D_x^{\alpha}\right)^n[f(x)]$is the$n$-th order α-fractionalderivative of $f(x)$. We note that $\left({}_{x_0}D_x^{\alpha}\right)^n\neq{}_{x_0}D_x^{n\alpha}$ in general. On the other hand, we define the fractional integral of $f(x)$, $\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\left({}_{x_0}D_x^{-\alpha}\right)[f(x)]$, where $\alpha>0$ and $f(x)$ is called α-fractional integrable function. We have the following property [22].

Proposition 2.2: *Let* α,β,c *be real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_0D_x^\alpha\right)\left[x^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \quad (2)$$

and

$${}_0D_x^\alpha[c] = 0. \quad (3)$$

Definition 2.3 ([23])**:** TheMittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)}, \quad (4)$$

where α is a real number, $\alpha > 0$, and z is a complex variable.

Definition 2.4 ([22])**:** Suppose that $0 < \alpha \le 1$ and x is a real variable. Then $E_\alpha(x^\alpha)$ is called α-fractional exponential function, and the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)}, \quad (5)$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}. \quad (6)$$

In the following, we introduce a new multiplication of fractional functions.

Definition 2.5 ([32])**:** If λ, μ, z are complex numbers, $0 < \alpha \le 1$, j, l, n are non-negative integers, and a_n, b_n are real numbers, $p_n(z) = \frac{1}{\Gamma(n\alpha+1)}z^n$ for all n. The $\otimes$ multiplication is defined by

$$p_j(\lambda x^\alpha) \otimes p_l(\mu y^\alpha) = \frac{1}{\Gamma(j\alpha+1)}(\lambda x^\alpha)^j \otimes \frac{1}{\Gamma(l\alpha+1)}(\mu y^\alpha)^l = \frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(\lambda x^\alpha)^j(\mu y^\alpha)^l, \quad (7)$$

where $\binom{j+l}{j} = \frac{(j+l)!}{j!\,l!}$.

Let $f(\lambda x^\alpha)$ and $g(\mu y^\alpha)$ be two fractional functions,

$$f(\lambda x^\alpha) = \sum_{n=0}^{\infty} a_n\, p_n(\lambda x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(\lambda x^\alpha)^n, \quad (8)$$

$$g(\mu y^\alpha) = \sum_{n=0}^{\infty} b_n\, p_n(\mu y^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(\mu y^\alpha)^n. \quad (9)$$

Then we define

$$f(\lambda x^\alpha) \otimes g(\mu y^\alpha) = \sum_{n=0}^{\infty} a_n\, p_n(\lambda x^\alpha) \otimes \sum_{n=0}^{\infty} b_n\, p_n(\mu y^\alpha)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} a_{n-m} b_m\, p_{n-m}(\lambda x^\alpha) \otimes p_m(\mu y^\alpha)\right). \quad (10)$$

Proposition 2.6: $f(\lambda x^\alpha) \otimes g(\mu y^\alpha) = \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\sum_{m=0}^{n}\binom{n}{m} a_{n-m} b_m\, (\lambda x^\alpha)^{n-m}(\mu y^\alpha)^m.$ (11)

Definition 2.7: Assume that $\left(f(\lambda x^\alpha)\right)^{\otimes n} = f(\lambda x^\alpha) \otimes \cdots \otimes f(\lambda x^\alpha)$ is the n times product of the fractional function $f(\lambda x^\alpha)$. If $f(\lambda x^\alpha) \otimes g(\lambda x^\alpha) = 1$, then $g(\lambda x^\alpha)$ is called the $\otimes$ reciprocal of $f(\lambda x^\alpha)$, and is denoted by $\left(f(\lambda x^\alpha)\right)^{\otimes -1}$.

Theorem 2.8(chain rule for fractional derivatives)([32])**:** *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g_\alpha(\mu x^\alpha) = \sum_{n=0}^{\infty} b_n\, p_n(\mu x^\alpha)$.

Let $f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) = \sum_{n=0}^{\infty} a_n\left(g_\alpha(\mu x^\alpha)\right)^{\otimes n}$ *and* $f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) = \sum_{n=1}^{\infty} n a_n\left(g_\alpha(\mu x^\alpha)\right)^{\otimes(n-1)}$, *then*

$$\left({}_0D_x^\alpha\right)\left[f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right)\right] = f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) \otimes \left({}_0D_x^\alpha\right)\left[g_\alpha(\mu x^\alpha)\right]. \quad (12)$$

Definition 2.9: If x, x_0 and a_n are real numbers, and $0 < \alpha \le 1$. The series $\sum_{n=0}^{\infty} a_n(x-x_0)^{n\alpha}$ is called a real α-fractional power series. Its disk of convergence intersects the real axis in an interval $(x_0 - r, x_0 + r)$ called the interval of convergence. Each real α-fractional power series defines a real valued sum function whose value at each x in the interval of convergence is given by

$$f(x^\alpha) = \sum_{n=0}^{\infty} a_n(x-x_0)^{n\alpha}. \quad (13)$$

This series is said to represent f in the interval of convergence, and it is called a α-fractional power series expansion of f about x_0.

Definition 2.10: Let $0 < \alpha \le 1$ and f be a real valued α-fractional function defined on an interval I in R.If f has α-fractional derivatives of every order at each point of I, we write $f \in C_\alpha^\infty(I)$. If $f \in C_\alpha^\infty(I)$ on some neighborhood of a point x_0, the series

$$\sum_{n=0}^{\infty}\frac{\left({}_{x_0}D_x^\alpha\right)^n[f(x)](x_0)}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \quad (14)$$

is called the α-fractional Taylor series about x_0 generated by f. To indicate that f generate this fractional Taylor series, we write

$$f(x^{\alpha}) \sim \sum_{n=0}^{\infty} \frac{\left({}_{x_0}D_x^{\alpha}\right)^n [f(x)](x_0)}{\Gamma(n\alpha+1)} (x-x_0)^{n\alpha}. \quad (15)$$

Theorem 2.11: *If* $0 < \alpha \leq 1$ *and* $f(x^{\alpha}) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n\alpha}$, *then*

$$f(x^{\alpha}) = \sum_{n=0}^{\infty} \frac{\left({}_{x_0}D_x^{\alpha}\right)^n [f(x)](x_0)}{\Gamma(n\alpha+1)} (x-x_0)^{n\alpha}. \quad (16)$$

Proof Since

$$\left({}_{x_0}D_x^{\alpha}\right)^n [f(x)](x_0) = \left({}_{x_0}D_x^{\alpha}\right)^n \left[\sum_{n=0}^{\infty} a_n (x-x_0)^{n\alpha}\right] = a_n \cdot \Gamma(n\alpha+1), \quad (17)$$

it follows that $a_n = \frac{\left({}_{x_0}D_x^{\alpha}\right)^n [f(x)](x_0)}{\Gamma(n\alpha+1)}$, and hence the desired result holds. q.e.d.

III. APPLICATIONS

In the following, we will give some examples to illustrate the applications of fractional Taylor series method provided in this paper.

Theorem 3.1: *Let* $0 < \alpha \leq 1$ *and* r *be a real number, then the* α*-fractional binomial series*

$$\left(1 + \frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes r} = \sum_{n=0}^{\infty} \frac{(r)_n}{\Gamma(n\alpha+1)} x^{n\alpha}, \quad (18)$$

and

$$\left(1 - \frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes r} = \sum_{n=0}^{\infty} \frac{(-1)^n (r)_n}{\Gamma(n\alpha+1)} x^{n\alpha}, \quad (19)$$

where $(r)_n = r(r-1)\cdots(r-n+1)$ *and* $-1 < \frac{1}{\Gamma(\alpha+1)} x^{\alpha} < 1$.

Proof Let $f(x^{\alpha}) = \left(1 + \frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes r}$. Then by chain rule for fractional derivatives, we obtain

$$\left({}_0D_x^{\alpha}\right)^n [f(x)](0) = r(r-1)\cdots(r-n+1) = (r)_n. \quad (20)$$

Therefore, by Theorem 2.11

$$\left(1 + \frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes r} = \sum_{n=0}^{\infty} \frac{(r)_n}{\Gamma(n\alpha+1)} x^{n\alpha}.$$

Similarly, we have

$$\left(1 - \frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes r} = \sum_{n=0}^{\infty} \frac{(-1)^n (r)_n}{\Gamma(n\alpha+1)} x^{n\alpha}. \quad \text{q.e.d.}$$

Remark 3.2: In particular, the α-fractional geometric series

$$\left(1 + \frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes -1} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\Gamma(n\alpha+1)} x^{n\alpha}, \quad (21)$$

and

$$\left(1 - \frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes -1} = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n\alpha+1)} x^{n\alpha} \quad (22)$$

for $-1 < \frac{1}{\Gamma(\alpha+1)} x^{\alpha} < 1$.

Theorem 3.3: *Assume that* $0 < \alpha \leq 1$, *then the* α*-fractional logarithmic function has the following fractional Taylor series expansion*

$$Ln_{\alpha}(x^{\alpha}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1\right)^{\otimes n} \quad (23)$$

for $-1 < \frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 < 1$.

Proof Since $\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$ for $|x-1| < 1$. It follows that

$$Ln_{\alpha}(x^{\alpha}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1\right)^{\otimes n}$$

for $-1 < \frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 < 1$. q.e.d.

Theorem 3.4: *Let* $0 < \alpha \leq 1$, *then the* α*-fractional arctan function*

$$arctan_{\alpha}(x^{\alpha}) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{\Gamma((2n+1)\alpha+1)} x^{(2n+1)\alpha}, \quad (24)$$

where $-1 < \frac{1}{\Gamma(2\alpha+1)} x^{2\alpha} < 1$.

ProofIf$-1 < \frac{1}{\Gamma(2\alpha+1)}x^{2\alpha} < 1$,then

$$arctan_\alpha(x^\alpha) = \left({}_0I_x^\alpha\right)\left[\left(1+\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\right)^{\otimes -1}\right]$$

$$= \left({}_0I_x^\alpha\right)\left[\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{\Gamma(2n\alpha+1)}x^{2n\alpha}\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n(2n)!\left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(2n\alpha+1)}x^{2n\alpha}\right]$$

$= \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{\Gamma((2n+1)\alpha+1)}x^{(2n+1)\alpha}$. q.e.d.

Theorem 3.5:*If* $0 < \alpha \leq 1$ *andmis a positive integer, then*

$$\frac{1}{\Gamma(m\alpha+1)}x^{m\alpha} \otimes E_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{\binom{n+m}{n}}{\Gamma((n+m)\alpha+1)}x^{(n+m)\alpha}, \quad (25)$$

$$\frac{1}{\Gamma(m\alpha+1)}x^{m\alpha} \otimes cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n+m}{2n}}{\Gamma((2n+m)\alpha+1)}x^{(2n+m)\alpha}, \quad (26)$$

and

$$\frac{1}{\Gamma(m\alpha+1)}x^{m\alpha} \otimes sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n+m+1}{2n+1}}{\Gamma((2n+m+1)\alpha+1)}x^{(2n+m+1)\alpha} \quad (27)$$

for any real numberx^α.

Proof$\frac{1}{\Gamma(m\alpha+1)}x^{m\alpha} \otimes E_\alpha(x^\alpha) = \frac{1}{\Gamma(m\alpha+1)}x^{m\alpha} \otimes \sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$

$$= \sum_{n=0}^{\infty}\frac{\binom{n+m}{n}}{\Gamma((n+m)\alpha+1)}x^{(n+m)\alpha}.$$

$$\frac{1}{\Gamma(m\alpha+1)}x^{m\alpha} \otimes cos_\alpha(x^\alpha) = \frac{1}{\Gamma(m\alpha+1)}x^{m\alpha} \otimes \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n+m}{2n}}{\Gamma((2n+m)\alpha+1)}x^{(2n+m)\alpha}.$$

$$\frac{1}{\Gamma(m\alpha+1)}x^{m\alpha} \otimes sin_\alpha(x^\alpha) = \frac{1}{\Gamma(m\alpha+1)}x^{m\alpha} \otimes \sum_{n=0}^{\infty}\frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n+m+1}{2n+1}}{\Gamma((2n+m+1)\alpha+1)}x^{(2n+m+1)\alpha}.$$

q.e.d.

IV. CONCLUSIONS

As mentioned above, by using a new multiplication, fractional Taylor series method andchain rule for fractional derivatives, the fractional Taylor series expansions of some fractional functionscan be obtained. In fact,the fractional Taylor series method iswidely used, which can easily solve many problems of fractional differential equations. In the future, we will take advantage of this method to expand our research fields to engineering mathematics and fractional calculus.

REFERENCES

[1]. F. Mainardi, *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*, World Scientific, 2010.
[2]. R. L. Magin, Fractional calculus in bioengineering, *13th International Carpathian Control Conference*, 2012.
[3]. G. M., Zaslavsky, *Hamiltonian chaos and fractional dynamics*, Oxford University Press, Oxford, 2005.
[4]. A. Carpinteri, F. Mainardi, (Eds.), *Fractals and fractional calculus in continuum mechanics*, Springer, Wien, 1997.
[5]. Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, *Advanced Engineering Technology and Application*, *5(2)*, 2016, 41-45.
[6]. Hasan, A. Fallahgoul, Sergio M. Focardi and Frank J. Fabozzi, *Fractional calculus and fractional processes with applications to financial economics*, Academic Press, 2017.
[7]. B. Carmichael, H. Babahosseini, SN Mahmoodi, M. Agah, The fractional viscoelastic response of human breast tissue cells, *Physical Biology*, *12(4)*, 2015, 046001.
[8]. N. Heymans, Dynamic measurements in long-memory materials: fractional calculus evaluation of approach to steady state, *Journal of Vibration and Control*, *14(9)*, 2008, 1587-1596.
[9]. V. E. Tarasov, Mathematical economics: application of fractional calculus, *Mathematics*, *8(5)*, 2020, 660.
[10]. T. Das, U. Ghosh, S. Sarkar, and S. Das, Time independent fractional Schrödinger equation for generalized Mie-type potential in higher dimension framed with Jumarie type fractional derivative, *Journal of Mathematical Physics*, *59(2)*, 2018, 022111.
[11]. S. Biswas, U. Ghosh, Approximate solution of homogeneous and nonhomogeneous $5\alpha^{th}$ order space-time fractional KdV, *International Journal of Computational Methods*, *18(1)*, 2021, 2050018.
[12]. C. -H. Yu, A study on fractional RLC circuit, *International Research Journal of Engineering and Technology*, *7(8)*, 2020, 3422-3425.
[13]. R. Barrio, M. Rodriguez, A. Abad, F. Blesa, Breaking the limits: the Taylor series method, *Applied Mathematics and Computation*, *217*, 2011, 7940-7954.
[14]. R. D. Neidinger, Directions for computing truncated multivariate Taylor series,*Mathematics of Computation*, *74(249)*, 2004, 321-340.
[15]. M. Razzaghi, M. Razzaghi, Solution of linear two-point boundary value problems withtime-varying coefficients via Taylor series,*International Journal of Systems Science*,*20(11)*, 1989, 2075-2084.
[16]. B. Bülbül, M. Sezer, A Taylor matrix method for the solution of a two-dimensional linearhyperbolic equation, *Applied Mathematics Letters*, *24(10)*, 2011, 1716-1720.
[17]. G. Groza, M. Razzaghi, A Taylor series method for the solution of the linear initial–boundary-value problems for partial differential equations, *Computers & Mathematics with Applications*, *66(7)*, 2013, 1329-1343.
[18]. Z. Odibat, N. Shawagfeh, Generalized Taylor's formula, *Applied Mathematics and Computation*, *186(1)*, 2007, 286-293.
[19]. J. J. Trujillo, M. Rivero, B. Bonilla, On a Riemann-Liouville generalized Taylor's formula, *Journal of Mathematical Analysis and Applications*,*231*, 1999, 255-265.
[20]. V. S. Krishnasamy, S. Mashayekhi, M. Razzaghi, Numerical solutions of fractional differential equations by using fractional Taylor basis, *IEEE/CAA Journal of AutomaticaSinica*, *4(1)*, 2017, 98-106.
[21]. G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, *Computers and Mathematics with Applications*,*51*, 2006, 1367-1376.
[22]. U. Ghosh, S. Sengupta, S. Sarkar, and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, *American Journal of Mathematical Analysis*, *3(2)*, 2015, 32-38.
[23]. J. C. Prajapati, Certain properties of Mittag-Leffler function with argument x^{α}, $\alpha > 0$,*Italian Journal of Pure and Applied Mathematics*, *30*, 2013, 411-416.
[24]. C. -H. Yu, Fractional Clairaut's differential equation and its application, *International Journal of Computer Science and Information Technology Research*, *8(4)*, 2020, 46-49.
[25]. C. -H. Yu, Separable fractional differential equations, *International Journal of Mathematics and Physical Sciences Research*, *8(2)*, 2020, 30-34.
[26]. C. -H. Yu, Integral form of particular solution of nonhomogeneous linear fractional differential equation with constant coefficients, *International Journal of Novel Research in Engineering and Science*, *7(2)*, 2020, 1-9.
[27]. C. -H. Yu, A study of exact fractional differential equations, *International Journal of Interdisciplinary Research and Innovations*, *8(4)*, 2020, 100-105.

[28]. C. -H. Yu, Research on first order linear fractional differential equations, *International Journal of Engineering Research and Reviews, 8(4)*, 2020, 33-37.
[29]. C. -H. Yu, Method for solving fractional Bernoulli's differential equation, *International Journal of Science and Research*, *9(11)*, 2020, 1684-1686.
[30]. C. -H. Yu, Using integrating factor method to solve some types of fractional differential equations, *World Journal of Innovative Research*, *9(5)*, 2020, 161-164.
[31]. C. -H. Yu, Two Types of Second Order Fractional Differential Equations, *2021 International Conference on Advances in Optics and Computational Sciences*, *Journal of Physics: Conference Series*, IOP Publishing, *1865*, 2021, 042138.
[32]. C. -H. Yu, Differential properties of fractional functions, *International Journal of Novel Research in Interdisciplinary Studies*, *7(5)*, 2020, 1-14.