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A Relation Between the Z – J Functions and The Elzaki and Sumudu Transform for Differential Equations with a Proposed Transformation

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Abstract: The following article is only a study and relation on the reduction functions or reduction function that helps to solve more quickly some Parabolic and Hyperbolic PDEs and which function is related to the Laplace Transform and Elzaki Sumudu, also the relation which saves the exponential function with the same solutions of each Equation.

Keyword: Laplace Transform – Elzaki and Sumudu Transform, Exponential Functions, Zenteno – Julia Functions, Transformation Functions

Introduction

From the topic discussed previously [7] we had reached some forms of reduction which are the following

Tabla 1

Linear Partial Differential Equation	Reduction Functions Zenteno – Julia	Reduced Equation
Inhomogeneous Hyperbolic Diffusion Equation $\frac{\partial^2\varphi}{\partial t^2}+\frac{\partial\varphi}{\partial t}-\frac{\partial^2\varphi}{\partial x^2}=f$	$\varphi(x,t)=Z(x,t)e^{-\frac{1}{2}t}+J(x,t)e^{\frac{1}{2}t}$ $\varphi(x,t)=Z(x,t)e^{\frac{1}{2}t}+J(x,t)e^{-\frac{1}{2}t}$	Inhomogeneous Wave Equation without J $\frac{\partial^2 z}{\partial t^2}=\frac{\partial^2 z}{\partial x^2}+fe^{\pm\frac{1}{2}t}$
Inhomogeneous Hyperbolic Diffusion Equation of the Telegraph $\frac{\partial^2\varphi}{\partial t^2}+\frac{\partial\varphi}{\partial t}-\frac{\partial^2\varphi}{\partial x^2}+\varphi=f$	$\varphi(x,t)=Z(x,t)e^{-\frac{1}{2}t}-J(x,t)e^{\frac{1}{2}t}$ $\varphi(x,t)=Z(x,t)e^{\frac{1}{2}t}-J(x,t)e^{-\frac{1}{2}t}$	Klein Gordon Equation $\frac{\partial^2 z}{\partial t^2}=a\frac{\partial^2 z}{\partial x^2}+\left(b+\frac{1}{4}k^2\right)z$
Hyperbolic Diffusion Equation $\frac{\partial^2\varphi}{\partial t^2}+\frac{\partial\varphi}{\partial t}-\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial\varphi}{\partial x}=0$	$\varphi(x,t)=Z(x,t)e^{\frac{1}{2}(x-t)}+J(x,t)e^{\frac{1}{2}(x+t)}$ $\varphi(x,t)=Z(x,t)e^{\frac{1}{2}(x+t)}+J(x,t)e^{\frac{1}{2}(x-t)}$	Homogeneous Wave Equation sin J $\frac{\partial^2 z}{\partial t^2}=\frac{\partial^2 z}{\partial x^2}$
Hyperbolic Equation of Advection Diffusion Inhomogeneous Reaction $\frac{\partial^2\varphi}{\partial t^2}+\frac{\partial\varphi}{\partial t}-\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial\varphi}{\partial x}=-\varphi+f$	$\varphi(x,t)=Z(x,t)e^{\frac{1}{2}(x-t)}-J(x,t)e^{\frac{1}{2}(x+t)}$ $\varphi(x,t)=Z(x,t)e^{\frac{1}{2}(x+t)}-J(x,t)e^{\frac{1}{2}(x-t)}$	Inhomogeneous Klein Gordon Equation $\frac{\partial^2 z}{\partial t^2}=a\frac{\partial^2 z}{\partial x^2}+a_t z+fe^{-\alpha x+\beta t}$
Advection Equation Diffusion Inhomogeneous Reaction $\frac{\partial\varphi}{\partial t}-\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial\varphi}{\partial x}=-\varphi+f$	Case where for reduction it can be used $\varphi(x,t)=Z(x,t)e^{\frac{1}{2}(x-t)}+J(x,t)e^{\frac{1}{2}x}$	Inhomogeneous Diffusion Equation $\frac{\partial z}{\partial t}=a\frac{\partial^2 z}{\partial x^2}+[-J_t+J_{xx}]e^{2\beta t}+fe^{-\alpha x+\beta t}$

We can see that the general function is

$$\boldsymbol{\varphi(x,y) = Z(x,y)e^{\alpha x+\beta t} + J(x,y)e^{\alpha x-\beta t}} \quad \textbf{(1)}$$

Now we can do the following, alpha and beta are real numbers, where alpha is large in order to reduce the desired expression, the function J is the inhomogeneous part like this

$$\varphi(x,y) = e^{\alpha x}(Z(x,y)e^{\beta t} + J(x,y)e^{-\beta t})$$
$$\varphi(x,y) = e^{\alpha x}(Z(x,y)e^{\beta t})$$
$$e^{-\alpha x}\varphi(x,t) = Z(x,t)e^{\beta t}$$

Both functions are equal and have the exponential reducing function, now taking the derivative with respect to t with x= 0 and we have to note that it is 1/β in all the previous expressions

First derivative

$$e^{-\frac{1}{\beta}t}\frac{\partial\varphi(x,t)}{\partial t} = \frac{Z}{\beta} + \frac{\partial Z}{\partial t}$$

Second derivative

$$\beta e^{-\frac{1}{\beta}t}\frac{\partial^2\varphi(x,t)}{\partial t^2} = \frac{Z}{\beta} + 2\frac{\partial Z}{\partial t} + \frac{\partial^2 Z}{\partial t^2}\beta$$

Third derivative

$$\beta e^{-\frac{1}{\beta}t}\frac{\partial^3\varphi(x,t)}{\partial t^3} = \frac{Z}{\beta^2} + 3\frac{1}{\beta}\frac{\partial Z}{\partial t} + 3\frac{\partial^2 Z}{\partial t^2} + \beta\frac{\partial^3 Z}{\partial t^3}$$

A polynomial arises that we can see how it follows

$$\beta e^{-\frac{1}{\beta}t}\frac{\partial\varphi(x,t)}{\partial t} = Z + \frac{\partial Z}{\partial t}\beta$$
$$\beta^2 e^{-\frac{1}{\beta}t}\frac{\partial^2\varphi(x,t)}{\partial t^2} = Z + 2\beta\frac{\partial Z}{\partial t} + \frac{\partial^2 Z}{\partial t^2}\beta^2$$
$$\beta^3 e^{-\frac{1}{\beta}t}\frac{\partial^3\varphi(x,t)}{\partial t^3} = Z + 3\beta\frac{\partial Z}{\partial t} + 3\beta^2\frac{\partial^2 Z}{\partial t^2} + \beta^3\frac{\partial^3 Z}{\partial t^3}$$

Which is related to the degree of the derivative by transforming it, now the part in x, we obtain a similar polynomial form. Now if Z= 0, from the beginning and deriving the left part again, we have

First derivative

$$e^{-\frac{1}{\beta}t}\frac{\partial\varphi(x,t)}{\partial t} = \varphi\left(\frac{1}{\beta}\right)e^{-\frac{1}{\beta}t}$$

Second derivative

substituting the results of the first derivative

$$\beta e^{-\frac{1}{\beta}t}\frac{\partial^2\varphi(x,t)}{\partial t^2} = 2\frac{\partial\varphi}{\partial t}e^{-\frac{1}{\beta}t} - \frac{\varphi}{\beta}e^{-\frac{1}{\beta}t}$$

$$e^{-\frac{1}{\beta}t}\frac{\partial^2\varphi(x,t)}{\partial t^2} = \frac{\varphi}{\beta^2}e^{-\frac{1}{\beta}t}$$

Third derivative

$$e^{-\frac{1}{\beta}t}\frac{\partial^3\varphi(x,t)}{\partial t^3} = \frac{\varphi}{\beta^3}e^{-\frac{1}{\beta}t}$$

Now let's integrate the left part of the first derivative

$$\beta e^{-\frac{1}{\beta}t}\frac{\partial\varphi(x,t)}{\partial t} = \beta\int_0^\infty e^{-\frac{1}{\beta}t}\frac{\partial\varphi(x,t)}{\partial t} = \beta\left(\left[\varphi e^{-\frac{1}{\beta}t}\right] + \frac{1}{\beta}\int_0^\infty \varphi e^{-\frac{1}{\beta}t}\,dt\right) = -\varphi(x,0)\beta + \frac{1}{\beta}T(\beta)$$

Where the initial condition at t = 0 must be substituted and we can see that the right part is also obtained.

$$\frac{1}{\beta}T(\beta) = \frac{1}{\beta}\int_0^\infty \varphi e^{-\frac{1}{\beta}t}\,dt = \varphi\left(\frac{1}{\beta}\right)e^{-\frac{1}{\beta}t} = \frac{1}{\beta}\int_0^\infty \varphi e^{-\frac{1}{\beta}t}\,dt$$

The First Derivative Transform is obtained. With the Second Derivative

$e^{-\frac{1}{\beta}t}\frac{\partial^2\varphi(x,t)}{\partial t^2} = \frac{\varphi}{\beta^2}e^{-\frac{1}{\beta}t}$	$\beta\int_0^\infty e^{-\frac{1}{\beta}t}\frac{\partial^2\varphi(x,t)}{\partial t^2}dt = \frac{1}{\beta^2}T(\beta) - \varphi(x,0) - \beta\frac{\partial\varphi(x,0)}{\partial t}$

Finding the same part again plus the initial conditions in time

The Third Derivative

$e^{-\frac{1}{\beta}t}\frac{\partial^3\varphi(x,t)}{\partial t^3} = \frac{\varphi}{\beta^3}e^{-\frac{1}{\beta}t}$	$\beta\int_0^\infty e^{-\frac{1}{\beta}t}\frac{\partial^3\varphi(x,t)}{\partial t^3}dt = \frac{1}{\beta^3}T(\beta) - \frac{\varphi(x,0)}{\beta} - \frac{\partial\varphi(x,0)}{\partial t} - \beta\frac{\partial^2\varphi(x,0)}{\partial t^2}$

Now let's see that we can put it like this, multiplying by 1/β, which is the term of the derivative in e

Primera Derivada

$$\frac{1}{\beta}\left(e^{-\frac{1}{\beta}t}\frac{\partial\varphi(x,t)}{\partial t}\right) = \frac{1}{\beta}\left(\frac{Z}{\beta} + \frac{\partial Z}{\partial t}\right)$$

Segunda Derivada $$\frac{1}{\beta}\left(e^{-\frac{1}{\beta}t}\frac{\partial^2\varphi(x,t)}{\partial t^2}\right)=\frac{1}{\beta}\left(\frac{Z}{\beta^2}+\frac{2}{\beta}\frac{\partial Z}{\partial t}+\frac{\partial^2 Z}{\partial t^2}\right)$$

Tercera Derivada $$\frac{1}{\beta}\left(e^{-\frac{1}{\beta}t}\frac{\partial^3\varphi(x,t)}{\partial t^3}\right)=\frac{1}{\beta}\left(\frac{Z}{\beta^3}+3\frac{1}{\beta^2}\frac{\partial Z}{\partial t}+\frac{3}{\beta}\frac{\partial^2 Z}{\partial t^2}+\frac{\partial^3 Z}{\partial t^3}\right)$$

Now by completing the first derivative, the left part and integrating we obtain the first part of the expression of the first derivative of a function of the Sumudu Transform

$$\frac{1}{\beta}\left(e^{-\frac{1}{\beta}t}\frac{\partial\varphi(x,t)}{\partial t}-\varphi\frac{1}{\beta}e^{-\frac{1}{\beta}}\right)=\frac{1}{\beta}\left(e^{-\frac{1}{\beta}t}\frac{\partial\varphi(x,t)}{\partial t}\right)=\frac{1}{\beta}\left(\varphi(x,t)\frac{1}{\beta}e^{-\frac{1}{\beta}}\right)$$

$$\frac{1}{\beta}\int_0^\infty\left(e^{-\frac{1}{\beta}t}\frac{\partial\varphi(x,t)}{\partial t}\right)dt=\frac{1}{\beta}\left(\frac{1}{\beta}\int_0^\infty\left(\varphi(x,t)e^{-\frac{1}{\beta}}dt\right)\right)=\frac{S[f(\beta)]}{\beta}$$

Note that with alpha = 0 we can eliminate from the spatial part of x

$$e^{-\alpha x}\varphi(x,t)=Z(x,t)e^{\beta t}$$
$$Z(x,t)=\varphi(x,t)e^{-\beta t}$$

$$Z(x,t)=\int_0^\infty\varphi(x,t)e^{-\beta t}\,dt \qquad (2)$$

And we have the Laplace Transform that can be related to these previous expressions.

Relating to the Elzaki and Sumudu Transform

Now from the properties of the Elzaki and Sumudu transform which is the form of transformation that is related to the normal definition

$$E[f(t),u)]=T(u)=u\int_0^\infty e^{\frac{-t}{u}}f(t)dt \qquad \mathbf{(3)}$$

With $u\ \in(k1,k2)$

$$|f(t)|<\begin{Bmatrix}Me^{\frac{-t}{k1}}\,t\le 0\\ Me^{\frac{t}{k2}}\,t\ge 0\end{Bmatrix}$$

With the derivative of n order of f with respect to t as

$$E[f(t)^n]=\frac{T(\beta)}{\beta^n}-\sum_{k=0}^{n-1}\beta^{2-n+k}\varphi^k(0)$$

From the previous and general function, with Z= J, constant functions M

$$\varphi(x,t)=Me^{-\frac{t}{\beta}}\ y\ \varphi(x,t)=Me^{\frac{t}{\beta}}$$

$$S[f(t),u)]=S(u)=\frac{1}{u}\int_0^\infty e^{\frac{-t}{u}}f(t)dt \qquad \mathbf{(4)}$$

With the derivative of n order of f with respect to t as [5]

$$S[f(t)^n]=\frac{S(\beta)}{\beta^n}-\sum_{k=0}^{n-1}\frac{\varphi^k(0)}{\beta^{n-k}}$$

The following definition is from [9].

Definition 1. The space W of exponential decay test functions is the space of complex valued functions φ(t) that satisfy the following properties:

(i) φ(t) is infinitely differentiable, i.e. φ(t) ∈ C^∞(R^n).

(ii) φ(t) and its derivatives of all orders vanish into infinity faster than the reciprocal of the exponential of order 1/ω; That's it

|e^(1/ω) D^kφ(t)| < M, ∀ 1/ω, k.

Then it is said that a function f(t) is of exponential growth if and only if f(t) together with all its derivatives grow more slowly than the exponential function of order 1/ω; that is, there exists a real constant 1/ω and M such that D^kφ(t) < M e^(1/ω). A linear continuous functional on the space W of test functions is called an exponentially growing distribution, and this dual space of W is denoted W.

Example 1

The following problem of a bar with the diffusion of heat

$$\frac{\partial\varphi(x,t)}{\partial t}=a^2\frac{\partial^2\varphi(x,t)}{\partial x^2}-A(\varphi-Ta)$$

With $\varphi(x,0)=0$and without border, using$\varphi(x,t)=Ze^{\frac{t}{\beta}}$the PDE transforms to

$$\frac{\partial Z(x,t)}{\partial t}=a^2\frac{\partial^2 Z(x,t)}{\partial x^2}+ATae^{-\frac{t}{\beta}}$$

With $\beta=-\frac{1}{A}$with solution

$$Z=\int_0^t\int_{-\infty}^{\infty}ATae^{-\frac{\tau}{\beta}}\left(\frac{e^{\frac{-(x-\varepsilon)^2}{4at}}}{2\sqrt{\pi at}}\right)d\varepsilon d\tau$$

with solution$Z=Ta(e^{At}-1)$ y $\varphi(x,t)=Ta(1-e^{-At})$

Making use of the expression below and substituting on each side and as the previous definition

$$\frac{\partial\varphi(x,t)}{\partial t}<(\frac{Z}{\beta}+\frac{\partial Z}{\partial t})e^{\frac{1}{\beta}t}$$

$$ATae^{-At}<Ae^{-At}$$

The pink curve is$ATae^{-At}$and the curve in purple isAe^{-At}

ATa = 0.5 and A = 1

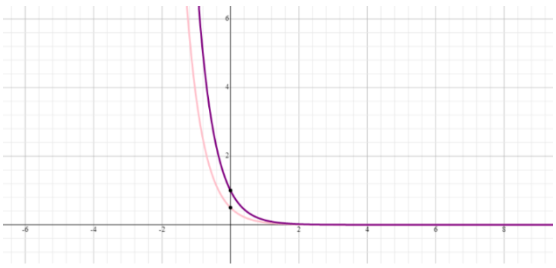

Figure 1

ATa = 0.05 y A = 1

Figure 2

Example 2

$$\frac{\partial\varphi(x,t)}{\partial t}=\frac{\partial^2\varphi(x,t)}{\partial x^2}$$

With x>0 and t >0 $\varphi(0,t)=1$ y $\varphi(x,0)=0$, with$\varphi(x,t)=Ze^{-\frac{t}{\beta}}$. Gives us the following expression

$$\frac{\partial Z(x,t)}{\partial t}=\frac{\partial^2 Z(x,t)}{\partial x^2}+\frac{Z}{\beta}$$

The term of Z/beta = 0 and with$\beta=1$thus the solution of the boundaryless Heat Equation is

$$Z(x,t)=Aerf\left(\frac{x}{2\sqrt{t}}\right)+B$$

Inserting the transformed initial conditions

$$Z(x,t)=e^{\frac{t}{\beta}}\left(-erf\left(\frac{x}{2\sqrt{t}}\right)+1\right)$$

With$\varphi(x,t)=Ze^{-\frac{t}{\beta}}$If we apply the Laplace Transform it is the same.

$$\varphi(x,t)=fcer\left(\frac{x}{2\sqrt{t}}\right)$$

Making use of the expression below and substituting on each side and as the previous definition

$$\frac{\partial\varphi(x,t)}{\partial t}<(-\frac{Z}{\beta}+\frac{\partial Z}{\partial t})e^{-\frac{1}{\beta}t}$$

$$-\frac{2e^{-\left(\frac{x}{2\sqrt{t}}\right)^2}}{\sqrt{\pi}}<fcer\left(\frac{x}{2\sqrt{t}}\right)$$

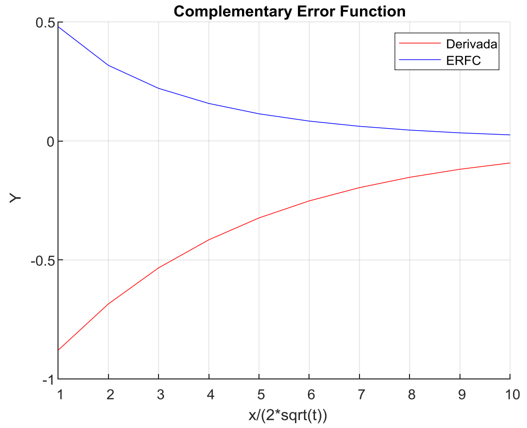

Figure3

Applying the Elzaki transformation, we get

$E\left[\frac{\partial\varphi(x,t)}{\partial t}\right] = \frac{T}{\beta} - \beta\varphi(x,0)$	(5)
$E\left[\frac{\partial^2\varphi(x,t)}{\partial x^2}\right] = \frac{d^2T(x,\beta)}{dx^2}$	(6)

So

$$\frac{d^2T(x,\beta)}{dx^2} = \frac{T}{\beta}$$

According to the general expression of the transformation a=b= f(β)=0 the solution of the ODE is

$$T(x,\beta) = C1e^{\sqrt{\frac{1}{\beta}}x} + C2e^{-\sqrt{\frac{1}{\beta}}x}$$

x > 0

$$T(x,\beta) = C2e^{-\sqrt{\frac{1}{\beta}}x}$$

C2 = β^2 and

$$T(x,\beta) = \beta^2 e^{-\sqrt{\frac{1}{\beta}}x}$$

Taking the Inverse Elzaki Transform [10]

$$E^{-1}\left[F(\nu)\right](t) = \frac{1}{2\pi \mathbf{j}}\int_{a-\mathbf{j}\infty}^{a+\mathbf{j}\infty} F\left(\frac{1}{\nu}\right) e^{t\nu}\, \nu\, d\nu = \sum \text{residues of } \left[F\left(\frac{1}{\nu}\right) e^{t\nu}\, \nu\right].$$

Thus, leaving the inverse as

$$E^{-1}[F(\beta)] = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{e^{-x\sqrt{\beta}}}{\beta} e^{\beta t}\, d\beta = fcer\left(\frac{x}{2\sqrt{t}}\right)$$

Now an observation if a

$$T(x,\beta) = \beta^2 e^{-\sqrt{\frac{1}{\beta}}x}$$

With

$$\frac{1}{\beta} = p$$

And knowing that beta = 1 of the previous value found, this is p = 1, we have the inverse form of Laplace

$$T(x,p) = e^{-\sqrt{p}x}$$

Now C2= 1/p according to the transformation that it has of the Laplace Transformation and thus

$$T(x,p) = \frac{e^{-\sqrt{p}x}}{p}$$

Without loss of generality, we have the following property

$$E\left[f(t)\right](\nu) = \nu\, \mathcal{L}\left[f(t)\right]\left(\frac{1}{\nu}\right) \quad \text{and} \quad \mathcal{L}\left[f(t)\right](\lambda) = \lambda\, E\left[f(t)\right]\left(\frac{1}{\lambda}\right).$$

Now we have the following with x = t we can see from the reduction functions that both parts, the x part and the t part can be equal, you can see the transformation between the ratio of alpha and beta

$$\varphi(x,t) = e^{\alpha x}\left(Z(x,t)e^{\beta t}\right) = e^{\alpha/\beta x+t}\left(Z(x,t)\right) = e^{\alpha/\beta t}(Z(x,t))e^{t}$$

$$e^{-\alpha/\beta t}e^{-t}\varphi(x,t) = Z(x,t)$$

Also if, Z' is the spatial part in x and constant as a parameter for a transformation now be $y = e^{-\alpha x}$ and$z = e^{-yt/\beta}$ deriving$\frac{dz}{dx} = \frac{dy}{dx}e^{-yt/\beta}$ is obtained$Z'e^{Z' t/\beta}$ either with alfa =1 and x=0 $\varphi(x,t) = Z(x,t)e^{-\alpha x}e^{t/\beta}$ =$Z(x,t)Z'e^{Z' t/\beta}$ now we can have this

$$\varphi(x,t)\frac{e^{-Z' t/\beta}}{Z'} = Z(x,t)$$

If alpha equals 1 and x=0 , Z' is 1 and respects the transformation with

$$\varphi(x,t)e^{-t/\beta} = Z(x,t)$$

We could also see that the beta accompanies the derivative in phi rising n times with that property, if we integrate from 0 to infinity and derive with respect to t the part of the Exponential is obtained

$$\frac{\beta e^{-Z' t/\beta}}{Z'^2} \text{ y } \frac{e^{-Z' t/\beta}}{Z'}$$

Now from Definition 1 then we have. **with**$\boldsymbol{\beta} \in (\boldsymbol{k1},\boldsymbol{k2})$

$$|\boldsymbol{f}(\boldsymbol{t})| < \begin{Bmatrix} \boldsymbol{M}\boldsymbol{e}^{\frac{-Z' t}{k1}} \; \boldsymbol{t} \leq \boldsymbol{0} \\ \boldsymbol{M}\boldsymbol{e}^{\frac{Z' t}{k2}} \; \boldsymbol{t} \geq \boldsymbol{0} \end{Bmatrix}$$

$$\boldsymbol{ZJ}[\boldsymbol{f}(\boldsymbol{t}),\boldsymbol{\beta})] = \boldsymbol{ZJ}(\boldsymbol{\beta}) = \frac{\boldsymbol{\beta}^{n}}{Z'}\int_{0}^{\infty} \boldsymbol{e}^{\frac{-Z' t}{\beta}}\boldsymbol{f}(\boldsymbol{t})\boldsymbol{dt} = \frac{\boldsymbol{\beta}^{n}}{Z'}\langle \boldsymbol{f}(\boldsymbol{t}),\boldsymbol{e}^{\frac{-Z' t}{\beta}}\rangle \quad \textbf{(4)}$$

$$\left|\frac{\beta^n}{Z'}\int_0^{\infty} e^{\frac{-Z' t}{\beta}}f(t)dt\right| < \frac{M\beta^n}{Z'}\int_0^{\infty} \boldsymbol{e}^{(\frac{-Z'}{\beta}-1/\omega)t}f(t)dt$$

The right hand side of the above expression is **Re** $\frac{\boldsymbol{\beta}^{n}}{Z'} > \boldsymbol{1}/\boldsymbol{\omega}$y $e^{\frac{-Z' t}{\beta}}f(t) \in W$

Taking Theorem 2 [14] with A = $(f(t)|\exists M,\tau_1,\tau_2 > 0\,, |f(t)| < Me^{t/\tau j}\; si\; t\; \in (-1)^j\; x\;[0,\infty))$

$\frac{\beta^n}{Z}\,and e^{\frac{-Z' t}{\beta}}\,with\; n = 1\; therefore\; it\; is\; had\; in\; beta\; as\; the - 1\; and\; so\; we\; can\; have\; as \frac{1}{Z}\frac{1}{\beta} = \frac{1}{Z}\left(\frac{1}{\eta} + i\frac{1}{\tau}\right)$ n is just a constant value for the transformation. Now taking in the Transformation, and multiplying by the exp of the definition

$$e^{-\frac{1}{Z\eta}-i\frac{1}{Z\tau}} = e^{-\frac{1}{Z\eta}}\left(\cos\left(\frac{t}{Z\tau}\right) - isen\left(\frac{t}{Z\tau}\right)\right)$$

$$\int_0^{\infty} f(x)\cos(\frac{t}{Z\tau})e^{-\frac{t}{Z\eta}}\,dt - i\int_0^{\infty} f(x)\text{sen}(\frac{t}{Z\tau})e^{-\frac{t}{Z\eta}}dt$$

$$\int_0^{\infty} |f(x)|\left|\cos(\frac{t}{Z\tau})\right|e^{-\frac{t}{Z\eta}}\,dt \leq M\int_0^{\infty} e^{\left(\frac{1}{K}-\frac{1}{Z\eta}\right)t}\,dt$$

We have evaluating the Integral on the right, just as the Imaginary part of the transformation exists now we know what Z is worth from the exponential to alpha with x=0 or alpha = 0, we have the same result like [14].

$$= \frac{MK\eta Z}{\eta Z - K}$$

Now we will see with the following ODE examples. With

$\frac{dy}{dt} + y = 0$y(0)=1

Using the previous transformation, we have

$$ZJ\left[\frac{dy}{dt}\right] = -y(x,0)\frac{\beta^n}{Z'} + \frac{Z'}{\beta}\hat{\varphi}$$

This is how we are left

$$\hat{\varphi} = \frac{\beta^{n+1}}{Z'(Z' + \beta)}$$

Taking the inverse we get like this, now taking s= $\beta Z'$we have directly the solution in Laplace

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{\beta Z' t}d\beta Z'}{(1-\beta Z')}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}ds}{(1-s)}= e^{-t}$$

Also if we use ds = zdβ we can see that the function complements itself and is also the solution, if alpha or x is equal to 0 only part in t is obtained

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}ds}{Z'(1-s)}=\frac{e^{-t}}{Z'}= e^{\alpha x-t}$$

Next example

$\frac{dy}{dt}+2y=t$ y(0)=1 with$ZJ[t]=\frac{\beta^n}{Z'^3}$ now in the ODE

$\left[-\frac{\beta^n}{Z'}+\frac{Z'}{\beta}\hat{\varphi}\right]+2\hat{\varphi}=\frac{\beta^n}{Z'^3}$ and$\hat{\varphi}=\frac{\beta^{n+3}}{Z'^3(Z'+2\beta)}+\frac{\beta^{n+1}}{Z'(Z'+2\beta)}$

Taking the inverse and taking s= $\beta Z'$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{\beta Z' t}d\beta Z'}{\beta^2 Z'^2(2-\beta Z')}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}ds}{s^2(2-s)}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{\beta Z' t}d\beta Z'}{(2+\beta Z')}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}ds}{(2+s)}$$

With solution and if we use again ds = zdβ

$$y(t) = -\frac{1}{4}H(t)+\frac{t}{2}+\frac{5}{4}e^{-2t}$$

$$y(t) = e^{\alpha x}\left[-\frac{1}{4}H(t)+\frac{t}{2}+\frac{5}{4}e^{-2t}\right]$$

$\frac{d^2y}{dt^2}+y=0$ y(0)=y'(0)=1 Using the transformation we have

$\hat{\varphi}=\frac{\beta^{n+2}}{Z'(\beta^2+Z'^2)}+\frac{\beta^{n+1}}{(\beta^2+Z'^2)}$now taking the Inverse and takings= $\beta Z'$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{\beta Z' t}d\beta Z'}{1+\beta^2 Z'^2}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}ds}{1+s^2}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\beta Z' e^{\beta Z' t}d\beta Z'}{1+\beta^2 Z'^2}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{s e^{st}ds}{1+s^2}$$

$$y(t) = sen(t)+\cos(t)$$

$\frac{d^2y}{dt^2}+4y=9t$y(0)=0 and y'(0)=7Using the transformation we have

$\hat{\varphi}=\frac{9\beta^{n+4}}{Z'^3(Z'^2+4\beta^2)}+\frac{7Z'\beta^{n+2}}{Z'^3(Z'^2+4\beta^2)}$now taking the Inverse and taking s= $\beta Z'$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{9e^{\beta Z' t}d\beta Z'}{\beta^2 Z'^2(\beta^2 Z'^2+4)}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{9e^{st}ds}{s^2(s^2+4)}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{7e^{\beta Z' t}d\beta Z'}{(\beta^2 Z'^2+4)}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}ds}{(s^2+4)}$$

$$y(t) = \frac{9t}{4}-\frac{38}{16}sen(2t)$$

$\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=0$ y(0)=-1 y y'(0)=3 using the transformation it is reduced and an algebraic equation of 2 degree is obtained based on beta and it is$(3\beta+Z')(\beta-Z')$

$\hat{\varphi}=-\frac{\beta^{n+1}}{Z'(3\beta+Z')}$now taking the Inverse and taking s= βZ'

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{\beta Z' t}d\beta Z'}{(\beta Z'+3)}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}ds}{(s+3)}$$

$$y(t) = e^{-3t}$$

$\frac{d^2y}{dt^2} + a^2\frac{dy}{dt} = f(t)$ con y(0)=0 y y'(0)=0 Using the transformation we have

$\hat{\varphi} = \frac{\beta^2 \overline{f(t)}}{Z'^2+a^2\beta Z'}$ now taking the inverse

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\beta^n Z' e^{\beta Z' t} d\beta Z'}{(\beta^2 Z'^2 + \beta Z' a^2)} = \frac{\beta^n Z'}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st} ds}{(s^2 + sa^2)}$$

As you can see the β^n Z' is not available, we can put n=0 and Z' we know that it is the other exponential which we can put x = 0, so by Residuals we obtain the solution and since it is Laplace it is a Convolution, we also have the following expression e^(-it) when indicating the sine, so we can put e^(-it)=1 and thus we have the solution

$$y(t) = \frac{1}{a}\int_0^x sen(a(x-t))f(t)dt$$

Thus, another variant or Transform linked to Laplace, Elzaki and Sumudu is proposed, among others such as the Natural, Aboodh, Kashuri – Fundo, Srivastava, ZZ, Ramadan Group [12] and the SEE Complex [13].

Then another variation of these Transformations is proposed, the Inverse obeys the Elzaki inverse.

ZJ Transform	**ZJ Inverse Transform**
$\boldsymbol{ZJ[f(t),\beta Z')] = ZJ(\beta Z') = \frac{\beta^n}{Z'}\int_0^\infty e^{\frac{-Z't}{\beta}} f(t)dt}$	$\boldsymbol{ZJ^{-1}[f(Z'\beta),t)] = \frac{\beta^n Z'}{2\pi i}\int_{-\infty}^{\infty} e^{Z'\beta t} f\left(\frac{1}{\beta}\right) d\beta Z'}$

Table of some transformations

1	$\frac{\beta^{n+1}}{z^2}$
t	$\frac{\beta^{n+2}}{z^3}$
$\sqrt{t}$	$\frac{\sqrt{\pi}\beta^{n+3/2}}{2z^{5/2}}$
e^{at}	$\frac{\beta^{n+1}}{z(z-a)}$
$sen(bt)$	$\frac{b\beta^{n+2}}{z(z^2+b^2\beta^2)}$
$\cos(bt)$	$\frac{\beta^{n+1}}{(z^2+b^2\beta^2)}$

Conclusions

In summary, we can see that the functions or function Z – J principal can be related to the Laplace transform and from the new transformations demonstrated by Elzaki and Sumudu, we can see that the Equation that reduces to the Partial Differential Equation has the relationship and is also You can use some of the transformations for your solution, satisfying definition 1, the Mathematical properties of such transformations are in the references for more essential details and applications.

The proposed transformation function also resolves the examples given, among others, and the change of s = zβ converts the inverse function to a Laplace function, obeying other transformations described above. Care must be taken as the last example where the betas are not available. to n, which can be easily maneuvered, if we leave Z as seen in the first examples, it is the spatial part of alpha in x showing the part of the transformation function of the exponentials.

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