Applications of the ZJ Transform for Differential Equations

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Abstract: The following article is a study, application and results on the ZJ Transform to solve Differential Equations and which the Transform is related to the Laplace, Elzaki and Sumudu Transform.
Keywords: Laplace Transform, Elzaki and Sumudu Transform, ZJ Transform

Introduction

Within the methods to solve Differential Equations there are many methods, as well as those of Transforms and among them is the Laplace Transform, one of the best known and most powerful tool, this variation of Transformation is born from the Relationship with the Laplace Transform, from Elzaki and Sumudu [1] in [2] we have obtained said variant of Transformation and in this the obtaining of the Inverse ZJ Transform is proposed with the following definition of [6]

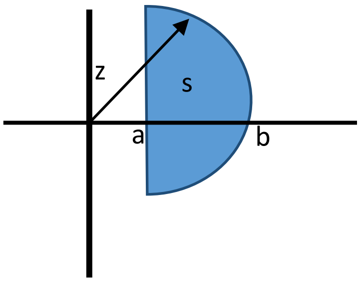
$$\boldsymbol{ZJ[f(t),\beta Z)] = ZJ(\beta Z) = \frac{\beta^n}{Z}\int_0^\infty e^{\frac{-Zt}{\beta}} f(t)dt} \quad (1)$$

Méthod

We briefly evaluate obtaining the Inverse Transform, evaluating by method of Residues, where the second Contour Integral must be 0

$$\frac{1}{2\pi i}\left[\int_{a+ib}^{a-ib} \frac{F\left(\frac{z}{\beta}\right)dz}{z-\left(\frac{z}{\beta}\right)} + \oint \frac{F\left(\frac{z}{\beta}\right)dz}{z-\left(\frac{z}{\beta}\right)}\right]$$

Now looking at the denominator by inequality of the triangle, the denominator is the distance between z and s = (z/β) on the contour C



Figure(1) in Plane

Taking the following and multiplying by both sides

$$\frac{1}{\left|z-\left(\frac{z}{\beta}\right)\right|} \le \frac{1}{b-\left|\frac{z}{\beta}\right|-a}$$

$$\frac{\left|F\left(\frac{z}{\beta}\right)\right|}{\left|z-\left(\frac{z}{\beta}\right)\right|} \le \frac{|z|^k m}{b^k\left(b-\left|\frac{z}{\beta}\right|-a\right)|\beta|^k}$$

The perimeter is $m\pi b$ of the contour and where $\left|F\left(\frac{z}{\beta}\right)\right| \le \frac{|z|^k m}{|\beta|^k}$ Also now we have for the contour C where $\left|F\left(\frac{z}{\beta}\right)\right| \le \frac{m}{|b|^k}$

$$\frac{\left|F\left(\frac{z}{\beta}\right)\right|}{\left|z-\left(\frac{z}{\beta}\right)\right|} \le \frac{|z|^k m\pi b}{b^k\left(b-\left|\frac{z}{\beta}\right|-a\right)|\beta|^k} \le \frac{|z|^k m\pi b}{b^{k+1}\left(1-\frac{\left|\frac{z}{\beta}\right|+a}{b}\right)|\beta|^k}$$

$$\frac{|z|^k m\pi}{b^k\left(1-\frac{\left|\frac{z}{\beta}\right|+a}{b}\right)|\beta|^k} = \lim_{b\to\infty}\frac{|z|^k m\pi}{b^k\left(1-\frac{\left|\frac{z}{\beta}\right|+a}{b}\right)|\beta|^k} = 0$$

So we only have left and ordering

$$\frac{1}{2\pi i}\left[\int_{a+ib}^{a-ib}\frac{F\left(\frac{z}{\beta}\right)dz}{z-\left(\frac{z}{\beta}\right)}\right] = \frac{1}{2\pi i}\left[\int_{a+ib}^{a-ib}\frac{\beta F\left(\frac{z}{\beta}\right)dz}{\beta z - z}\right]$$

$$\frac{1}{2\pi i}\left[\int_{a+ib}^{a-ib}\beta F\left(\frac{z}{\beta}\right)e^{(\beta z-z)t}d(\beta z-z)\right] = \frac{1}{2\pi i}\left[\int_{a+ib}^{a-ib}\beta F\left(\frac{z}{\beta}\right)e^{(\beta z)t}d\beta z - \int_{a+ib}^{a-ib}\beta F\left(\frac{z}{\beta}\right)e^{(-z)t}dz\right]$$

Since z is constant thedz = 0

$$\frac{1}{2\pi i}\left[\int_{a+ib}^{a-ib}\beta F\left(\frac{z}{\beta}\right)e^{(\beta z)t}d\beta z\right] = \frac{1}{2\pi i}\left[\beta^n z\int_{a+ib}^{a-ib}F\left(\frac{1}{\beta}\right)e^{(\beta z)t}d\beta z\right]$$

$$\boldsymbol{\frac{1}{2\pi i}\beta^n z\int_{a+ib}^{a-ib}F\left(\frac{1}{\beta}\right)e^{(\beta z)t}d\beta z} \qquad (2)$$

And according to Elzaki the Inverse Transformation respects the following form [7]

Relating to the Elzaki and Sumudu Transform

Now from the properties of the Elzaki and Sumudu transform which is the form of transformation that is related to the normal definition

$$\boldsymbol{E[f(t),u)] = T(u) = u\int_0^\infty e^{\frac{-t}{u}}f(t)dt} \qquad (3)$$

With$u \in (k1, k2)$

$$|f(t)| < \begin{Bmatrix} Me^{\frac{-t}{k1}}\ t \le 0 \\ Me^{\frac{t}{k2}}\ t \ge 0 \end{Bmatrix}$$

With the derivative of n order of f with respect to t as

$$\boldsymbol{E[f(t)^n] = \frac{T(\beta)}{\beta^n} - \sum_{k=0}^{n-1}\beta^{2-n+k}\varphi^k(0)}$$

From the previous and general function, with Z= J, constant functions M

$$\varphi(x,t) = Me^{-\frac{t}{\beta}}\ y\ \varphi(x,t) = Me^{\frac{t}{\beta}}$$

$$\boldsymbol{S[f(t),u)] = S(u) = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}}f(t)dt} \qquad (4)$$

With the derivative of n order of f with respect to t

$$\boldsymbol{S[f(t)^n] = \frac{S(\beta)}{\beta^n} - \sum_{k=0}^{n-1}\frac{\varphi^k(0)}{\beta^{n-k}}}$$

The following definition is from [3], which validates the definition and the obtaining of these transformations.

Definition 1. The space W of exponential decay test functions is the space of complex valued functions φ(t) that satisfy the following properties:

(i) φ(t) is infinitely differentiable, i.e. φ(t) $\in C^\infty(R^n)$.

(ii) φ(t) and its derivatives of all orders vanish into infinity faster than the reciprocal of the exponential of order 1/ω; that's

$|e^{1/\omega} D^k \varphi(t)| < M, \forall\ 1/\omega, k.$

Then it is said that a function f(t) is of exponential growth if and only if f(t) together with all its derivatives grow more slowly than the exponential function of order 1/ω; that is, there exists a real constant 1/ω y M such thatD^kφ(t) < M $e^{1/\omega}$. A linear continuous functional on the space W of test functions is called an exponentially growing distribution, and this dual space of W is denoted W.

Examples of Application and Discussion

Now we will see with the following ODE examples

Example 1

$\frac{dy}{dt} + y = 0$**Withy(0)=1**

Using the previous transformation, we have

$$ZJ\left[\frac{dy}{dt}\right] = -y(x,0)\frac{\beta^n}{Z} + \frac{Z}{\beta}\hat{\varphi}$$

This is how we are left

$$\hat{\varphi} = \frac{\beta^{n+1}}{Z(Z+\beta)}$$

Taking the inverse of the previous Expression$f(\frac{1}{\beta})$ y por $z\beta^n$ we have directly the solution in Laplace

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{\beta Zt}\,d\beta Z}{(1-\beta Z)} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}\,ds}{(1-s)} = e^{-t}$$

Also if we use ds = zdβ we can see that the function complements itself and is also the solution, if alpha or x is equal to 0 only part in t is obtained

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}\,ds}{Z(1-s)} = \frac{e^{-t}}{Z} = e^{\alpha x - t}$$

Example 2

$\frac{dy}{dt} + 2y = t$**Withy(0)=1** with$ZJ[t] = \frac{\beta^n}{Z^3}$ now in the ODE

$\left[-\frac{\beta^n}{Z} + \frac{Z}{\beta}\hat{\varphi}\right] + 2\hat{\varphi} = \frac{\beta^n}{Z^3}and\hat{\varphi} = \frac{\beta^{n+3}}{Z^3(Z+2\beta)} + \frac{\beta^{n+1}}{Z(Z+2\beta)}$

Taking the inverse of the previous Expression$f(\frac{1}{\beta})$ and by$z\beta^n$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{\beta Zt}\,d\beta Z}{\beta^2 Z^2(2-\beta Z)} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}\,ds}{s^2(2-s)}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{\beta Zt}\,d\beta Z}{(2+\beta Z)} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}\,ds}{(2+s)}$$

With solution and if we use againds = zdβ where z = $e^{-\alpha x}$

$$y(t) = -\frac{1}{4}H(t) + \frac{t}{2} + \frac{5}{4}e^{-2t}$$

$$y(t) = e^{\alpha x}\left[-\frac{1}{4}H(t) + \frac{t}{2} + \frac{5}{4}e^{-2t}\right]$$

Example 3

$\frac{d^2y}{dt^2} + y = 0$**Withy(0)=y'(0)=1**Using the transformation we have

$$ZJ\left[\frac{d^2y}{dt^2}\right] = -\frac{\beta^n}{z}y'(x,0) - \frac{\beta^n}{\beta}y(x,0) + \frac{z^2}{\beta^2}\hat{\varphi}$$

$\hat{\varphi} = \frac{\beta^{n+2}}{Z(\beta^2+Z^2)} + \frac{\beta^{n+1}}{(\beta^2+Z^2)}$ Now taking the Inverse of the previous Expression$f(\frac{1}{\beta})$ and by$z\beta^n$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{\beta Zt}\,d\beta Z}{1+\beta^2Z^2}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}\,ds}{1+s^2}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\beta Ze^{\beta Zt}\,d\beta Z}{1+\beta^2Z^2}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{se^{st}\,ds}{1+s^2}$$

$$y(t)=\ sen(t)+\cos(t)$$

Example 4

$\frac{d^2y}{dt^2}+4y=9t$ **Withy(0)=0 and y'(0)=7**Using the transformation we have

$$ZJ\left[\frac{d^2y}{dt^2}\right]=-\frac{\beta^n}{z}y'(x,0)-\frac{\beta^n}{\beta}y(x,0)+\frac{z^2}{\beta^2}\hat{\varphi}$$

$\hat{\varphi}=\frac{9\beta^{n+4}}{Z'^3(Z'^2+4\beta^2)}+\frac{7Z'\beta^{n+2}}{Z'^3(Z'^2+4\beta^2)}$ Now taking the Inverse of the previous Expression$f(\frac{1}{\beta})$andtaking s= $\beta Z'$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{9e^{\beta Z't}d\beta Z}{\beta^2Z^2(\beta^2Z^2+4)}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{9e^{st}ds}{s^2(s^2+4)}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{7e^{\beta Zt}\,d\beta Z}{(\beta^2Z^2+4)}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}\,ds}{(s^2+4)}$$

$$y(t)=\frac{9t}{4}-\frac{38}{16}sen(2t)$$

Example 5

$\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=0$**With y(0)=-1 and y'(0)=3**using the transformation it is reduced and an algebraic equation of 2 degree is obtained based on beta and it is$(3\beta+Z)(\beta-Z)$

$\hat{\varphi}=-\frac{\beta^{n+1}}{z(3\beta+z)}$ Now taking the Inverse and taking s= βZ

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{\beta Z't}d\beta Z}{(\beta Z+3)}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{st}ds}{(s+3)}$$

$$y(t)=\ e^{-3t}$$

Example 6

$\frac{d^2y}{dt^2}+a^2\frac{dy}{dt}=f(t)$**Withy(0)=1 and y'(0)= -2**Using the transformation we have

$\hat{\varphi}=\frac{\beta^2\widehat{f(t)}}{z^2+a^2\beta^2}+\frac{z\beta^{n+1}-2\beta^{n+2}}{z(z^2+a^2\beta^2)}$ Now taking the Inverse of the previous Expression$f(\frac{1}{\beta})$ and by$z\beta^n$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\beta^nZ\,\widehat{f(t)}e^{\beta Zt}\,d\beta Z}{\beta^2z^2+a^2}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\widehat{f(t)}e^{st}\,ds}{(s^2+a^2)}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\beta Z\,e^{\beta Zt}\,d\beta Z}{\beta^2z^2+a^2}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{se^{st}ds}{(s^2+a^2)}$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{2e^{\beta Zt}\,d\beta Z}{\beta^2z^2+a^2}=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}-\frac{2e^{st}ds}{(s^2+a^2)}$$

We have a convolution solution as well

$$y(t)=cosat-\frac{2senat}{a}+\frac{1}{a}\int_0^t sen(a(x-u))f(u)du$$

Example 7

$\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=0$**Withy(0)=1 and y'(0)=0** Using the transformation it is reduced and we obtain

$$\hat{\varphi}=\frac{z\beta^{n+1}-\beta^{n+2}}{z(z-2\beta)(z+\beta)}$$

Now taking the Inverse of the previous Expression $f(\frac{1}{\beta})$ and by$z\beta^n$

$$\hat{\varphi} = \frac{z\beta - 1}{(\beta z - 2)(\beta z + 1)}$$

Taking s= βz

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{z\beta - 1\, e^{\beta Zt}\, d\beta Z}{(\beta z - 2)(\beta z + 1)} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{(s-1)\, e^{st}\, ds}{(s-2)(s+1)}$$

WithSolution

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

So then, as seen in the previous examples, another variant or Transform linked to Laplace, Elzaki and Sumudu is proposed, among others such as Natural, Aboodh, Kashuri - Fundo, Srivastava, ZZ, Ramadan Group [4] and the SEE Complex. [2]. The change of s = zβ converts the inverse function to a Laplace one obeying other transformations described above, if we leave Z as seen in the first examples it is the spatial part of alpha in x, showing the part of the transformation function of the exponentials.

ZJ Transform

$$\boldsymbol{ZJ[f(t), \beta Z)] = ZJ(\beta Z) = \frac{\beta^n}{Z}\int_0^{\infty} e^{\frac{-Zt}{\beta}} f(t)dt}$$

InverseTransform

$$\boldsymbol{ZJ^{-1}[f(Z\beta), t)] = \frac{\beta^n Z}{2\pi i}\int_{-\infty}^{\infty} e^{Z\beta t} f\left(\frac{1}{\beta}\right) d\beta Z}$$

Table 1 of sometransformations

1	$\frac{\beta^{n+1}}{z^2}$
t	$\frac{\beta^{n+2}}{z^3}$
$\sqrt{t}$	$\frac{\sqrt{\pi}\beta^{n+3/2}}{2z^{5/2}}$
e^{at}	$\frac{\beta^{n+1}}{z(z-a)}$
$sen(bt)$	$\frac{b\beta^2}{\beta^n z + z^2\beta^2}$
$cos(bt)$	$\frac{b\beta}{\beta^n z + b^2\beta^2}$
$t^n e^{kt}$	$\frac{n!\,\beta^n}{\left(k - \frac{z}{\beta}\right)^{n+1} z}$
$senh(bt)$	$\frac{b\beta^{n+2}}{z(z^2 - z^2\beta^2)}$
$cosh(bt)$	$\frac{\beta^{n+1}}{(z^2 - z^2\beta^2)}$
$\delta(t - t_0)$	$\frac{\beta^n e^{-\frac{z}{\beta}t0}}{z}$
$e^{at}F(t)$	$f(\frac{z}{\beta} - a)$
$\frac{dy}{dt}$	$-y(x,0)\frac{\beta^n}{z} + \frac{z}{\beta}\hat{\varphi}$
$\frac{d^2y}{dt^2}$	$-\frac{\beta^n}{z}y'(x,0) - \frac{\beta^n}{\beta}y(x,0) + \frac{z^2}{\beta^2}\hat{\varphi}$
$\frac{d^3y}{dt^3}$	$-\frac{\beta^n}{z}y''(x,0) - \frac{\beta^n}{\beta}y'(x,0) - \frac{\beta^n z}{\beta^2}y(x,0) + \frac{z^3}{\beta^3}\hat{\varphi}$

Specialforms of the particular solutions of thedifferentialequation of 1 order and 2 order

$\frac{dy}{dt} + py = f(t)$ **con y(0)=a**with

$$\hat{\varphi} = \frac{f(\hat{t}) + \frac{a\beta^n}{z}}{\frac{z}{\beta} + p}$$

Withthesameexample

$\frac{dy}{dt} + y = 0$ **withy(0)=1** Thus

$\hat{\varphi} = \frac{\frac{\beta^n}{z}}{\frac{z}{\beta}+1}$takingtheinverse$f\left(\frac{1}{\beta}\right)$youhave$\frac{1}{z\beta^n(z\beta+1)}$nowfor$z\beta^n$wehave$\frac{1}{(z\beta+1)}$ and like s= zβ youhavethesolution as$\frac{1}{(s+1)}$whichise^{-t}

$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = f(t)$**withy(0)=a y'(0)=b with particular solution**

$$\hat{\varphi} = \frac{f(\hat{t}) + \frac{b\beta^n}{z} + \frac{a\beta^n}{\beta} + pa\frac{\beta^n}{z}}{\frac{z^2}{\beta^2} + p\frac{z}{\beta} + q}$$

$\frac{d^2y}{dt^2} + y = 0$ **withy(0) = y'(0)=1**

$\hat{\varphi} = \frac{\frac{\beta^n}{z}+\frac{\beta^n}{\beta}}{\frac{z^2}{\beta^2}+q}$takingtheinverse$f\left(\frac{1}{\beta}\right)$youhave$\frac{1+z\beta}{z\beta^n(z^2\beta^2+1)}$nowfor$z\beta^n$wehave$\frac{1+z\beta}{(z^2\beta^2+1)}$

s= zβyouhavethesolution as$\frac{1+s}{(s^2+1)}$whichis$y(t) = sen(t) + \cos(t)$

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