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On Analysis of Existence and Uniqueness of a Deterministic Model

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Abstract. This paper presents adeterministic model for SEIVR epidemic model. The purpose is to examine the existence and uniqueness of our model. Picard *theorem is used toestablishthat there exists a solution and such solution is unique.*

1. Introduction

Malaria remains one of the most important human diseases throughout the tropical and subtropical regions of the world and causes more than 300million acute illness and at least one million deaths annually [14]. 90% of deaths due to malaria occur in young children. The search for a malaria vaccine is now over seventy years old [14]. Despite considerable efforts over the last three decades, and millions of dollars spent, there is still no registered vaccine against Plasmodium falciparum malaria. However, recent breakthroughs in malaria vaccines have given new hope that a safe, effective malaria vaccine may be found [16].

Mathematical models including vaccination aid in deciding on a vaccination strategy and in determining changes in qualitative behavior that could result from such a control measure [18]. In this paper, we formulated a mathematical model for a disease-modifying malaria vaccine that consider an individual who is susceptible and become exposed after coming in contact with an infected mosquito. The exposed class E reduces in population after the infection has manifested to join the infective class I, who later move to recovery class R.

1.1 The model Equations

2 The total population $N(t)$ is divided into four compartments with $N(t)=S(t)+E(t)+I(t)+V(t)+R(t)$, where S is the number of individuals in the susceptible class, E is the number of people who areexposed(the latent period, in which the person is infected but not yet infectious) but not vaccinated, I is the number of people who are infectious but not vaccinated, V is the number of individuals who are vaccinated and R is the immune class. This model is called an SEIVRS model. The differential equations for this model are:

$$\frac{dS}{dt}=\psi+\sigma R-\lambda S-dS \quad (1)$$

$$\frac{dE}{dt}=\lambda S-\alpha E-dE \quad (2)$$

$$\frac{dI}{dt}=\alpha E-\beta I-dI \quad (3)$$

$$\frac{dV}{dt}=\beta I-\varepsilon V--dV \quad (4)$$

$$\frac{dR}{dt}=\varepsilon V-\sigma R-dR \quad (5)$$

2. Basic Properties of the model

It can easily be shown that all the state variables of model remain non-negative for all non-negative initial conditions.
Consider the biological feasible region

$$\Sigma = \left\{ (S, E, I, V, R) \in \Re^5 : N \to \frac{\psi}{d} \right\}$$

2.1. Lemma1. The close Σ is positively invariant and attracting

Proof

Summing up (1) – (5), yield the rate of the total population

$$\frac{dN}{dt} = \psi - dN$$

Thus, the total hostpopulation (N) is bounded by $\frac{\psi}{d}$, so that $\frac{dN}{dt} = 0$ whenever

$N(t) = \frac{\psi}{d}$. It can easily been seen that

$N(t) = \frac{\psi}{d} + \left(N_0 - \frac{\psi}{d} \right) e^{-dt}$. In particular,

$N(t) = \frac{\psi}{d}$, if $N(0) = \frac{\psi}{d}$.

Hence, the region Σ is positively invariant and attracts all solutions in $\Re^5{}_+$.

3. Existence and Uniqueness of solution for the Model

For the mathematical model to predict the future of the system from its current state at time t_0, the initial value problem (IVP)

$$x^1 = f(t, x), \qquad x(t_0) = x_0 \qquad (6)$$

Must have a solution that exist and also unique.

In this subsection, we shall establish conditions for the existence and uniqueness of solution for the model of equations. Let

$$\begin{aligned} f_1(t, x) &= \psi + \sigma r - \lambda s - ds \\ f_2(t, x) &= \lambda s - \alpha e - de \\ f_3(t, x) &= \alpha e - \beta i - di \\ f_4(t, x) &= \beta i - \varepsilon v - dv \\ f_5(t, x) &= \varepsilon v - \sigma r - dr \end{aligned} \qquad (7)$$

So that

$$x^1 = f(t, x) = f(x) \qquad (8)$$

3.1. Definition1. Suppose function $f(t,x)$ has domain D in (t,x)-space and suppose there exists a constant $k>0$ such that if $(t,x^1),(t,x^2)\in D$ then

$$\left|f(t,x^1)-f(t,x^2)\right|\le k\left|x^1-x^2\right| \tag{9}$$

Then f satisfies a Lipchitz condition with respect to x in D, and k is a Lipchitz constant for f.

3.2. Theorem1. Let D be an open set in (t,x)-space. Let $(t_o,x^o)\in D$ and let a,b be positive constants such that the set

$$R=\left\{(t,x)\middle|\,|t-t_o|\le a, \left|x-x^o\right|\le b\right\} \tag{10}$$

is contained in D. Suppose function f is defined and continuous on D and satisfies a Lipchitz condition with respect to x in R. Let

$$Max=\max_{(t,x)\in R}\left|f(t,x)\right| \tag{11}$$

$$A=\min\left[a,\tfrac{b}{M}\right] \tag{12}$$

Then the differential equation

$$x^1=f(t,x) \tag{13}$$

has a unique solution $x(t,t_o,x^o)$ on (t_o-A,t_o+A) such that $x(t_o,t_o,x^o)=x^o$. This solution $x(t,t_o,x^o)$ is such that $\left|x(t,t_o,x^o)-x\right|\le MA$ for all $t\in(t_o-A,t_o+A)$. (14)

3.3. Lemma1. If $f(,x)$ has a continuous partial derivative $\dfrac{\partial f_i}{\partial f_j}$ on a bounded closed convex domain R, then it satisfies a Lipchitz condition in R.

We are interested in the region

$$1\le\varepsilon\le R\ . \tag{15}$$

We look for a bounded solution of the form

$$0<R<\infty. \tag{16}$$

We shall prove the following existence theorem.

3.4. Theorem2. Let D denote the region defined in (9) such that (15) and (16) hold. Then there exists a solution of model (7) which is bounded in the region D.

Proof. Let

$$f_1 = \psi + \sigma r - \lambda s - ds$$
$$f_2 = \lambda s - \alpha e - de$$
$$f_3 = \alpha e - \beta i - di \qquad (17)$$
$$f_4 = \beta i - \varepsilon v - dv$$
$$f_5 = \varepsilon v - \sigma r - dr$$

It suffices to show that

$\frac{\partial f_i}{\partial f_j}$, $i, j = 1,2,3,4,5$ are continuous.

$$\frac{\partial f_1}{\partial s} = \psi + \sigma r, \quad \left|\frac{\partial f_1}{\partial s}\right| = |\psi + \sigma r| < \infty, \quad \frac{\partial f_i}{\partial f_j}$$

$$\frac{\partial f_i}{\partial e} = 0, \qquad \left|\frac{\partial f_1}{\partial e}\right| = |0| < \infty,$$

$$\frac{\partial f_1}{\partial i} = 0, \qquad \left|\frac{\partial f_1}{\partial i} = |0|\right| < \infty,$$

$$\frac{\partial f_1}{\partial v} = 0, \qquad \left|\frac{\partial f_1}{\partial v}\right| = |0| < \infty,$$

$$\frac{\partial f_1}{\partial r} = 0, \qquad \left|\frac{\partial f_1}{\partial r}\right| = |0| < \infty,$$

Also, $$\frac{\partial f_2}{\partial s} = \lambda, \qquad \left|\frac{\partial f_2}{\partial s}\right| = |\lambda| < \infty,$$

$$\frac{\partial f_2}{\partial e} = -\alpha - d, \quad \left|\frac{\partial f_2}{\partial e}\right| = |-\alpha - d| < \infty,$$

$$\frac{\partial f_2}{\partial i} = 0, \qquad \left|\frac{\partial f_2}{\partial i}\right| = |0| < \infty,$$

$$\frac{\partial f_2}{\partial v} = 0, \qquad \left|\frac{\partial f_2}{\partial v}\right| = |0| < \infty,$$

$$\frac{\partial f_2}{\partial r} = 0, \qquad \left|\frac{\partial f_2}{\partial r}\right| = |0| < \infty,$$

$$\frac{\partial f_3}{\partial s} = 0, \qquad \left|\frac{\partial f_3}{\partial s}\right| = |0| < \infty,$$

$$\frac{\partial f_3}{\partial e} = \alpha, \qquad \left|\frac{\partial f_3}{\partial e}\right| = |\alpha| < \infty,$$

$$\frac{\partial f_3}{\partial i} = -\beta - d, \qquad \left|\frac{\partial f_3}{\partial i}\right| = |-\beta - d| < \infty,$$

$$\frac{\partial f_3}{\partial v} = 0, \qquad \left|\frac{\partial f_3}{\partial v}\right| = |0| < \infty,$$

$$\frac{\partial f_3}{\partial r} = 0, \qquad \left|\frac{\partial f_3}{\partial r}\right| = |0| < \infty$$

$$\frac{\partial f_{4i}}{\partial s} = 0, \qquad \left|\frac{\partial f_4}{\partial s}\right| = |0| < \infty,$$

$$\frac{\partial f_4}{\partial e} = 0, \qquad \left|\frac{\partial f_4}{\partial e}\right| = |0| < \infty,$$

$$\frac{\partial f_4}{\partial i} = \beta, \qquad \left|\frac{\partial f_4}{\partial i}\right| = |\beta| < \infty,$$

$$\frac{\partial f_4}{\partial v} = -\varepsilon - d, \qquad \left|\frac{\partial f_4}{\partial v}\right| = |-\varepsilon - d| < \infty,$$

$$\frac{\partial f_{4i}}{\partial r} = 0, \qquad \left|\frac{\partial f}{\partial r}\right| = |0| < \infty.$$

$$\frac{\partial f_5}{\partial s} = 0, \qquad \left|\frac{\partial f_5}{\partial s}\right| = |0| < \infty,$$

$$\frac{\partial f_5}{\partial e} = 0, \qquad \left|\frac{\partial f_5}{\partial e}\right| = |0| < \infty,$$

$$\frac{\partial f_5}{\partial i} = 0, \qquad \left|\frac{\partial f_5}{\partial i}\right| = |0| < \infty,$$

$$\frac{\partial f_5}{\partial v} = \varepsilon, \qquad \left|\frac{\partial f_5}{\partial v}\right| = |\varepsilon| < \infty,$$

$$\frac{\partial f_5}{\partial r} = -\sigma - d, \qquad \left|\frac{\partial f_5}{\partial r}\right| = |-\sigma - d| < \infty.$$

4.Conclusion

Since all these partial derivatives are continuous and bounded, therefore, by theorem (2), there exists a unique solution of (7) in the region D

References

[1]. H.W. Hethcote, The mathematics of infectious diseases, Society for Industrial and Applied Mathematics SIAM Review, 42, No. 4 (2000), 599-653.

[2]. J.Tumwiine, J.Y.T. Mugisha, L.S. Luboobi, A host-vector model for malaria with infective immigrants, Journal of Mathematical Analysis and Applications: 361 (2010) 139-149.

[3]. J.Tumwiine, J.Y.T. Mugisha, L.S. Luboobi, A mathematical model for the dynamics of malaria in a human host and mosquito vector with temporary immunity, Applied Mathematics and Computation 189 (2007) 1953 – 1965

[4]. P. Johansson, J. Leander, Mathematical Modeling of Malaria, Chalmers University of Technology Gothenburg, 2010.

[5]. N. Chitnis, J.M. Cushing and J.M. Hyman, Bifurcation Analysis of a Mathematical Model for Malaria Transmission. SIAM J.APPL.MATH.Vol. 67, No. 1, pp. 24 – 45.

[6]. J. Cronin, Differential Equations: Introduction and Qualitative Theory, Marcel Dekker, Inc. (1994).

[7]. L. Bubniakova,. The Mathematics of Infectious Diseases, M.sc thesis (2007).

[8]. J.K. Hale, Ordinary Differential Equations, John Wiley, New York, 1969.

[9]. G. Li, W. Wang, Z. Jin, Global stability of an SEIR epidemic model with constant immigration, Chaos Solutions Fractals 30 (4) (2006) 1012 – 1019

[10]. G. A. Ngwa, W. S. Shu, A mathematical model for endemic malaria with variable human and mosquito populations, Mathematics and Computer Modelling 32 (2000) 747 – 763.

[11]. World Health Organization, World malaria report 2005. Geneva. World Health Organization. WHO/HAM/MAL/2005.1102, 2005.

[12]. World Health Organization. Malaria – A global crisis, Geneva, 2000.

[13]. C.C. Carlos, W, Carlos, C. Garsow and Y, Abdul-Aziz. Mathematical models of Isolation and Quarantine, American Medical Association 290(2003) 21.

[14]. J.S. Robert. Could low-efficacy malaria vaccines increase secondary infections in endemic areas?

[15]. D.Lacitignola, B.Buonomo. On the backward bifurcation of a vaccination model with nonlinear incidence. Nonlinear Analysis: Modelling and Control, 16:1(2011):30-46.

[16]. S.Thomas, K. F.Gerry, M.Nicolas, R. Amanda, L.Molineaux, T. Fabrizio, H, Guy, U, Jurg, K, Dietz and M. Tanner. Mathematical modeling of the impact of malaria vaccines on the clinical epidemiology and natural history of Plasmodium Falciparum malaria: Overview. A.m.Trop.Med. Hyg. 75(2), 2006, pp. 1-10.

[17]. M.De la Sen, S. Alonso-Quesada and A. Ibeas. Observer-Based Vaccination for a SEIR Epidemic Model with Time-Varying Total Population.International Conference on Circuits, System and Simulation, IPCSIT, 7(2011).

[18]. J.Arino, C. Connell, MCCLUSKEY, and P.V.D. Driessche. Global result for an epidemic model with vaccination that exhibits backward bifurcation. SIAM J. APPL. MATH, 64:1 :(2003), 260-276.