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The Double ZJ Transform and its Convergence

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Abstract: The following article is a study and application about the double ZJ transform and its convergence to solve Differential Equations and Integral Equations too.

Summary: The article is a study and application about the double ZJ transform its convergence and its properties to solve Differential Equations and Integral Equations.

Keywords: Double Laplace Transform, Double Integral Transform, PDE, IE, Double Sumudu Transform, Double Elzaki Transform, Double ZJ Transform

1. Introduction

Now we will see the convergence and validation of the double ZJ transform. It must be said that the demonstration follows the same way as the double Laplace, Sumudu, Elzaki or Aboodh transform, except for the way in which the complex number beta was defined.

Integral transforms are important to deal with the solutions of differential equations subject to boundary conditions. The integral transform is a mathematical operator that produces a new function f(s) by integrating the product of an existing function $F(x)$ and a so-called kernel function $K\ (x,\ y)$ between suitable limits when doing the integration.

The process, which is called a transformation, is symbolized by the equation f(s) = $\int (x, y)F(x)dx$. In the Laplace transform, the kernel is $e(-$s$x)$ and the limits of integration are zero and plus infinity, in the Fourier transform, the kernel is e$(-ixy)$ and the limits are minus and plus infinity. When this happens, the above integral converges and if the limit does not exist, the integral diverges and there is no transform defined for f, the integral is the ordinary (improper) Riemann integral and the parameter s belongs to some domain on the real line or the complex plane.

We consider functions on the set A, defined by

$$A = \{f(t)\ |\exists M, \tau 1, \text{ and /or } \tau 2 > 0, \text{ such that } |\ f(t)| < Me^{\frac{|t|}{\tau j}},\ if\ t \in (-1)j \times [0, \infty)\}$$

For a given function on the set A, the constant M must be finite, while $\tau 1$ and $\tau 2$ need not exist simultaneously, and each can be infinite.

Instead of being used as a power of the exponential as in the case of the Laplace transform, in the Sumudu transform the variable u is used to factor the variable at t into the argument of the function f, just as in other transforms.

Let us look at the examples for the case of $f\ (t)$ in A, the Sumudu transform is defined by

$$S[f(t)] = \begin{Bmatrix} \int_0^\infty f(ut)e^{-t}dt & 0 \le u < \tau 2 \\ \int_0^\infty f(ut)e^{-t}dt & -\tau 1 \le u < 0 \end{Bmatrix}$$

In the Elzaki transform is defined for the exponential order function, it is considered a function on the set S, as in A defined as

$$S=\{f(t)\text{: } \exists M,\ k1\ ,k2>0,|\ f(t)|<Me^{\frac{|t|}{kj}}, if\ t \in (-1)j \times [0,\infty)\ \}$$

The Elzaki transform denoted by the operator E is defined as

$$E[f(t)] = T(v) = v\int_0^\infty f(t)e^{-vt}dt\ \ ,t > 0$$

The variable v in this transformation is used to factor the variable t into the argument of the function f.

Now the ZJ transform is defined for the exponential order function and on a set S as $\boldsymbol{\beta} \in (\boldsymbol{k1}, \boldsymbol{k2})$

$$|f(t)| < \left\{ \begin{array}{l} Me^{\frac{-zt}{k1}} \; t \leq 0 \\ Me^{\frac{zt}{k2}} \; t \geq 0 \end{array} \right\}$$

$$\boldsymbol{\beta} = \{ f(t) \colon \exists M, k1, k2 > 0, |f(t)| < Me^{\frac{|zt|}{kj}}, if \; t \in (-1)j \times [0,\infty) \}$$

$$ZJ[f(t)] = ZJ\left(\frac{z}{\beta}\right) = \frac{\beta^n}{z} \int_0^\infty f(t) e^{\frac{-zt}{\beta}} dt \quad , t > 0$$

$$= \frac{\beta^n}{z} \langle f(t), e^{\frac{-zt}{\beta}} \rangle$$

Now the variable $\frac{\boldsymbol{\beta^n}}{\boldsymbol{z}}$ in this transformation is used to factor the variable t into the argument of the function f, with z and n as integer constants. The purpose of this paper is to show the applicability of this new transformation and its efficiency in applying some convergence theorems.

With the following notations as $s = \frac{\beta}{z} and \frac{1}{s} = \frac{z}{\beta}$ now for $\frac{z1}{\beta 1} = \frac{z1}{p1} = \frac{z1}{p}$ y $\frac{z2}{\beta 2} = \frac{z2}{q1} = \frac{z1}{q}$ thus$\beta 1 = p$ and$\beta 2 = q$ and$s = q, \; po = \beta_1 or \; , qo = \beta_2 o$

2. Theorem of Convergence of the Double ZJ Integral

In this section, we prove the theorem of convergence of the double ZJ integral, which follows the same treatment as in the proof of the double integral of Elzaki and Sumudu

Teorem 2.1: Let $\phi(x,$y) be a function of two continuous variables in the positive quadrant of the xy plane. If the integral converges to p=po and q= qo then the integral converges to β_1<po, β_2< qo and

$$\frac{\beta 1^n}{z1} \frac{\beta 2^n}{z2} \int_0^\infty \phi(x,y) e^{-\frac{z1x}{\beta 1} - \frac{z2y}{\beta 2}} dxdy$$

With the following **Lemma 2.2** if the Integral

$$\frac{\beta 2^n}{z2} \int_0^\infty \phi(x,y) e^{-\frac{z2y}{\beta 2}} dy$$

converges at q= qo then the integral converges for β_2< qo, now the proof is as follows with

$$\alpha(x,y) = \frac{qo^n}{z2} \int_0^y \phi(x,u) e^{-\frac{z2u}{qo}} du$$

With α(x,0)=0 if y=0 and the limit of α when y tends to infinity exists, because

$$\frac{\beta 2^n}{z2} \int_0^\infty \phi(x,y) e^{-\frac{z2y}{\beta 2}} dy$$

Converges in β_2= qo, Now doing the following $\frac{\partial \alpha}{\partial y} = \frac{qo^n}{z2} \phi(x,u) e^{-\frac{z2y}{qo}}$ in the interval $0 < \epsilon < R$ Let R be infinite, clearing $\phi(x,u) = \frac{\partial \alpha}{\partial y} \left(\frac{Z2}{qo^n}\right) e^{\frac{z2y}{qo}}$ Now substituting in the previous integral as $\frac{\beta 2^n}{z2} \int_\epsilon^R \phi(x,y) e^{-\frac{z2y}{\beta 2}} dy$ and integrating by parts we have

$$= \frac{\beta 2^n}{qo^n} \left[\left(\frac{qo - \beta 2}{\beta 2 qo} \right) \int_0^\infty \alpha(x,y) e^{-\left(\frac{qo-\beta 2}{\beta 2 qo}\right) z2y} dy \right]$$

Now we need to see the limit of this, so now, using the "limit test" for convergence (Widder, 2005). For this we have

$$\frac{\beta 2^n}{qo^n} \left(\frac{qo-\beta 2}{\beta 2 qo} \right) \lim_{y\to\infty} y^2 \alpha(x,y) e^{-\left(\frac{qo-\beta 2}{\beta 2 qo}\right) z2y} = \lim_{y\to\infty} y^2 e^{-\left(\frac{qo-\beta 2}{\beta 2 qo}\right) z2y} \lim_{y\to\infty} \alpha(x,y) = 0 * \lim_{y\to\infty} \alpha(x,y) =$$

$$0, is \; finite \; and \; thus \; the \; Integral \; converges \frac{\beta 2^n}{z2} \int_0^\infty \phi(x,y) e^{-\frac{z2y}{\beta 2}} dy \; in \beta_2 < \text{q}o$$

Now a Lemma 2.3 if the previous Integral as $h(x, \beta_2) = \frac{\beta 2^n}{z2} \int_0^\infty \phi(x,y) e^{-\frac{z2y}{\beta 2}} dy$ converge forβ_2< q$o$and now the integral

$$= \frac{\beta 1^n}{z1}\int_0^\infty h(x,\beta_2)e^{-\frac{z1x}{\beta 1}}dx$$

Converge in $\beta 1 = po$ and this integral converge for $\beta 1 < po$. the demostratión is the same that lemna 2.2. Thisthe integral

$$\frac{\beta 1^n}{z1}\int_0^\infty h(x,\beta_2)e^{-\frac{z1x}{\beta 1}}dx$$

converge for $\beta 1 < po$, so the proof of the Teorem is as follows

$h(x,\beta_2) = \frac{\beta 2^n}{z2}\int_0^\infty \phi(x,y)e^{-\frac{z2y}{\beta 2}}dy$ converge for $\beta_2 <$ qo for the Lemma 2.2, now for **the Lemma 2.3** the integral $\frac{\beta 1^n}{z1}\int_0^\infty h(x,\beta_2)e^{-\frac{z1x}{\beta 1}}dx$ converge for $\beta 1 < po$, thus the Integral

$$\frac{\beta 1^n}{z1}\frac{\beta 2^n}{z2}\int_0^\infty \phi(x,y)e^{-\frac{z1x}{\beta 1}-\frac{z2y}{\beta 2}}dxdy \ converge\ for$$

$\beta 1 < po$, $\beta_2 <$ qo and $\beta 1 = p$, q= $\beta 2$ with

$$\frac{po^n}{z1}\frac{qo^n}{z2}\int_0^\infty \phi(x,y)e^{-\frac{z1x}{po}-\frac{z2y}{qo}}dxdy$$

The demonstration is complete. With two respective Corollaries

Corollary 2.4: if the integral $\frac{\beta 1^n}{z1}\frac{\beta 2^n}{z2}\int_0^\infty \phi(x,y)e^{-\frac{z1x}{\beta 1}-\frac{z2y}{\beta 2}}dxdy$ diverge for $p = po$ and q= qo so the integral $\frac{\beta 1^n}{z1}\frac{\beta 2^n}{z2}\int_0^\infty \phi(x,y)e^{-\frac{z1x}{\beta 1}-\frac{z2y}{\beta 2}}dxdy$ diverges for $\beta 1 < po$, $\beta_2 <$ qo.

Corollary 2.5: The region of convergence of the integral $\frac{\beta 1^n}{z1}\frac{\beta 2^n}{z2}\int_0^\infty \phi(x,y)e^{-\frac{z1x}{\beta 1}-\frac{z2y}{\beta 2}}dxdy$. is the positive quadrant of the xy plane. We now prove the absolute convergence of the integral $\frac{\beta 1^n}{z1}\frac{\beta 2^n}{z2}\int_0^\infty \phi(x,y)e^{-\frac{z1x}{\beta 1}-\frac{z2y}{\beta 2}}dxdy$

Teorem 2.6: If the integral $\frac{\beta 1^n}{z1}\frac{\beta 2^n}{z2}\int_0^\infty \phi(x,y)e^{-\frac{z1x}{\beta 1}-\frac{z2y}{\beta 2}}dxdy$ absolutely converges in p=po , q= qo then the integral converges absolutely for $\beta 1 < po$, $\beta_2 <$ qo.

The proof is

$$e^{\frac{-z1x}{po}-\frac{z2y}{qo}}|\phi(x,y)| \le e^{\frac{-z1x}{p}-\frac{z2y}{q}}$$

For $p \le po < \infty \ \ and \ \ q \le qo < \infty$ thus

$$\frac{po^n}{z1}\frac{qo^n}{z2}\int_0^\infty \phi(x,y)e^{-\frac{z1x}{po}-\frac{z2y}{qo}}dxdy \ \le \ |\phi(x,y)|\frac{p^n}{z1}\frac{q^n}{z2}\int_0^\infty \phi(x,y)e^{-\frac{z1x}{p}-\frac{z2y}{q}}dxdy$$

By hypothes is the integral on the right converges immediately we have that the integral on the left also converges in $\beta 1 < po$, $\beta_2 <$ qo. Therefore, the integral $\frac{\beta 1^n}{z1}\frac{\beta 2^n}{z2}\int_0^\infty \phi(x,y)e^{-\frac{z1x}{\beta 1}-\frac{z2y}{\beta 2}}dxdy$ absolutely converges for $\beta 1 < po$, $\beta_2 <$ qo.

3. Now for Uniform Convergence the Double Convergence of the Transformation will be Demonstrated

Teorem 3.1: If f (x, t) is continuous in [0, ∞) × [0, ∞) and

$$\frac{po^n}{z1}\frac{qo^n}{z2}\int_0^x\int_0^y f(u,v)e^{-\frac{z1u}{po}-\frac{z2v}{qo}}dudv$$

is limited in [0, ∞) × [0, ∞), then twice the ZJ transform of f converges uniformly to [p, ∞) × [q, ∞) if $p < po$, q < qo. For the demonstration we will use the following lemmas:

Lemma 3.2: If $g(x,t) = \frac{qo^n}{z2}\int_0^y f(x,v)e^{-\frac{z2v}{qo}}dv$ is limited in [0, ∞) then the ZJ Transform of f with respect to qo converges uniformly to [qo, ∞) if q < qo.

Demonstration: If 0 < r <∞then we consider 0 ≤ r ≤ r1 en $\frac{\beta2^n}{z2}\int_r^{r1} f(x,y)e^{-\frac{z2y}{\beta2}}dy$ multiplied and equal to$\frac{\beta2^n}{z2}\int_r^{r1} f(x,y)e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2y}e^{-\frac{z2y}{qo}}dy = \frac{\beta2^n}{qo}\int_r^{r1}\frac{\partial g(x,y)}{\partial y}e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2y}dy$ Now integrating by parts we have

$$= \frac{\beta2^n}{qo}\left[g(x,r1)e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2r1} - g(x,r)e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2r} + \left(\frac{qo-\beta2}{\beta2qo}\right)z2\int_r^{r1} e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2y}dy\right]$$

Therefore, if | (x, y) | ≤ M so

$\left|\frac{\beta2^n}{z2}\int_r^{r1} f(x,y)e^{-\frac{z2y}{\beta2}}dy\right| \leq \frac{\beta2^n}{qo}\left[g(x,r1)e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2r1} + g(x,r)e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2r} + \left(\frac{qo-\beta2}{\beta2qo}\right)z2\int_r^{r1} e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2y}dy\right] = M\frac{\beta2^n}{qo}\left(e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2r1} + e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2r} - e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2r1} + e^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2r}\right) = \frac{\beta2^n}{qo}\left[2Me^{-\left(\frac{qo-\beta2}{\beta2qo}\right)z2r}\right]$ for q<qo is convergent for some r in 0 ≤ r ≤ r1 by **the Cauchy criterion for uniform convergence**, reference

$$\frac{\beta2^n}{z2}\int_r^{r1} f(x,y)e^{-\frac{z2y}{\beta2}}dy$$

converges uniformly in [q, ∞) if q < qo. Therefore, the ZJ transform of f with respect to q converges uniformly to [q, ∞) if q < qo.

Lemma 3.3: If the integral $g(x,$ q) =

$$\frac{\beta2^n}{z2}\int_r^{r1} f(x,y)e^{-\frac{z2y}{\beta2}}dy$$

converges uniformly in [q, ∞) if q < qo and$\alpha(x,q) = \frac{\beta1^n}{z1}\int_0^x g(u,q)e^{-\frac{z1u}{po}}du$ is bounded on [0, ∞), then the ZJ Transform of f with respect to q converges uniformly on [p, ∞) if $p< po$.

The proof is similar to **Lemma 3.2**. Proving **Theorem 3.1 by Lemma 3.2**, the ZJ transform of f with respect to s converges uniformly on [q, ∞) if q < qo. Also **by Lemma 3.3**, the ZJ transform of g with respect to p converges uniformly on [p, ∞) if $p< po$.

Hence, the double ZJ transform of f converges uniformly on [p, ∞) × [q, ∞) if $p< po$, q < qo as seen in the previous proofs.

We now prove the differentiability of the double ZJ transform as well as in [Idrees MI, Ahmed Z, Awais M, and Perveen Z (2018).]

Teorem 3.4: if f (x, t) is continuous in [0, ∞) × [0, ∞) and

$$H(x,y) = \frac{po^n}{z1}\frac{qo^n}{z2}\int_0^x\int_0^y f(u,v)e^{-\frac{z1u}{po}-\frac{z2v}{qo}}dudv$$

It is bounded in [0, ∞) × [0, ∞) then the double ZJ transform of f is infinitely differentiable with respect to p and q in [p, ∞) × [q, ∞) if p <po , q < qo , with

$$\frac{\partial^{m+n} f\left(\frac{z1}{\beta1},\frac{z2}{\beta2}\right)}{\partial\left(\frac{z1}{\beta1}\right)^m\left(\frac{z2}{\beta2}\right)^n} = \frac{\partial^{m+n} f(p,q)}{\partial p^m q^n} = (-1)^{m+1}\frac{\beta1^n}{z1}\frac{\beta2^n}{z2}\int_0^\infty\int_0^\infty f(x,y)x^m y^n e^{-\frac{z1x}{\beta1}-\frac{z2y}{\beta2}}dxdy$$

$$= (-1)^{m+1}\frac{q^n}{z2}\frac{p^n}{z1}\int_0^\infty\int_0^\infty f(x,y)x^m y^n e^{-\frac{z1x}{p}-\frac{z2y}{q}}dxdy$$

For the proof we will use the following lemmas, as in the previous demonstrations.

Lemma 3.5: If $g(x,$ y) = $\frac{qo^n}{z2}\int_0^y f(x,v)e^{-\frac{z2v}{qo}}dv$ is bounded on [0, ∞) then the ZJ transform of f is infinitely differentiable with respect to q on [q, ∞) if q < qo with

$$\frac{\partial^n f\left(x,\frac{z_2}{\beta_2}\right)}{\partial\left(\frac{z2}{\beta_2}\right)^n}=\frac{\partial^n f(x,q)}{\partial q^n}=(-1)^n\frac{\beta_2{}^n}{z_2}\int_0^\infty f(x,y)y^n e^{-\frac{z2y}{\beta2}}dy$$

First, we will show that the integrals, where n is $\frac{\beta2^n}{z2}$ independent.

$$In\left(x,\frac{z_2}{\beta_2}\right)=(-1)^n\frac{\beta_2{}^n}{z_2}\int_0^\infty f(x,y)y^n e^{-\frac{z2y}{\beta2}}dy \quad with\, n=0,1,2,3\ldots$$

all converge uniformly on [q, ∞) if q < qo and if $0\le r\le r1$, then

$$\frac{\beta2^n}{z2}\int_r^{r1} f(x,y)y^n e^{-\frac{z2y}{\beta2}}dy=\frac{\frac{\beta2^n}{z2}}{\frac{\beta_{20}^2}{z2}}\int_r^{r1}\frac{\partial g}{\partial y}(x,y)y^n e^{-\left(\frac{\frac{\beta_{20}^2}{z2}-\frac{\beta2^n}{z2}}{\frac{\beta2^n\beta_{20}^2}{z2\ z2}}\right)y}dy$$

Remember that $\frac{\beta2^n}{\beta_{20}^2}=\frac{q}{qo}$ thus

$$=\frac{q}{qo}\left[g(x,r1)r1^n e^{-\left(\frac{qo-q}{qqo}\right)r1}-g(x,r)r^n e^{-\left(\frac{qo-q}{qqo}\right)r}-\int_r^{r1}\frac{d}{dy}\left(e^{-\left(\frac{qo-q}{qqo}\right)y}y^n\right)g(x,y)dy\right]$$

Therefore, if | g(x, y) | ≤ M < ∞ in [0, ∞) so

$$\left|\frac{\beta2^n}{z2}\int_r^{r1} f(x,y)y^n e^{-\frac{z2y}{\beta2}}dy\right|\le M\frac{q}{qo}\left[r1^n e^{-\left(\frac{qo-q}{qqo}\right)r1}+r^n e^{-\left(\frac{qo-q}{qqo}\right)r}-r1^n e^{-\left(\frac{qo-q}{qqo}\right)r1}+r^n e^{-\left(\frac{qo-q}{qqo}\right)r}\right]$$

$$\left|\frac{\beta2^n}{z2}\int_r^{r1} f(x,y)y^n e^{-\frac{z2y}{\beta2}}dy\right|\le\frac{2Mq}{qo}r^n e^{-\left(\frac{qo-q}{qqo}\right)r}$$

for $0\le r\le r1$

Now by the Cauchy criterion for uniform convergence (Trench, 2012). In (x, q) converges uniformly on [q, ∞) if q < qo. Now, using Trench (2012) and the induction proof in $\frac{\partial^n f(x,q)}{\partial q^n}=(-1)^n\frac{\beta2^n}{z2}\int_0^\infty f(x,y)y^n e^{-\frac{z2y}{\beta2}}dy$
It is noted that the ZJ transform of f is infinitely differentiable with respect to q in [q, ∞) if q < qo.
By induction it can be seen that $\frac{\partial f(x,q)}{\partial q}=-\frac{\beta2^n}{z2}\int_0^\infty f(x,y)ye^{-\frac{z2y}{\beta2}}dy$ is valid, now for $\frac{\partial^k f(x,q)}{\partial q^k}=(-1)^k\frac{\beta2^n}{z2}\int_0^\infty f(x,y)y^k e^{-\frac{z2y}{\beta2}}dy$ with a positive integer k it is also, now for $\frac{\partial^{k+1}f(x,q)}{\partial q^{k+1}}=(-1)^{k+1}\frac{\beta2^n}{z2}\int_0^\infty f(x,y)y^{k+1}e^{-\frac{z2y}{\beta2}}dy$ Also, you just have to differentiate k+1 times and see that it respects the previous notation.

Lemma 3.6: If the integral $\emptyset(x,q)=\frac{\beta2^n}{z2}\int_0^\infty y^n f(x,y)e^{-\frac{z2y}{\beta2}}dy$ converges uniformly in [q, ∞) if q < qo and h(x, q)= $\frac{\beta1o^n}{z1}\int_0^x \emptyset(x,q)e^{-\frac{z1u}{\beta1o}}dx$ is bounded on [0, ∞), then the ZJ transform of ∅ is infinitely differentiable with respect to p on [p, ∞) if $p< po$, with the $\frac{\partial^m\emptyset(x,q)}{\partial q^m}=(-1)^m\frac{\beta2^n}{z2}\int_0^\infty \emptyset(x,q)y^m e^{-\frac{z2y}{\beta2}}dx$ The proof is similar to

Lemma 3.5. Thus the proof of **Theorem 3.4** is as follows where $g(u$, y)= $\frac{qo^n}{z2}\int_0^y f(u,v)e^{-\frac{z2v}{qo}}dv$ is limited in [0, ∞).

By Lemma 3.5, the ZJ transform of f is infinitely differentiable with respect to q on [q, ∞) if q < qo. Also by Lemma 3.6, the ZJ transform of g is infinitely differentiable with respect to p on [p, ∞) if $p< po$. Therefore, the double ZJ transform of f is infinitely differentiable with respect to p and q on [p, ∞) × [q, ∞) if q < qo, $p<po$.

4. The Double ZJ Transform of a Double Integral is Now

Teorem 4.1 If$ZJ_2D[f(x,y)] = f(p,q)$ y $g(x,y) = \int_0^x \int_0^y f(u,v)dudv$ then the double ZJ Transform is of the Integral

$$ZJ_2\left[\int_0^x\int_0^y f(u,v)dudv\right] = \frac{\beta 1}{z1}\frac{\beta 2}{z2}\hat{\varphi} \quad or \frac{\beta_1^{n+1}}{z_1^2}\frac{\beta_2^{n+1}}{z_2^2}\hat{\varphi}$$

Remember that there are the$\frac{\beta 2^n}{z2}$, Now let's try the Inverse and now doing $f\left(\frac{1}{\beta 1\beta 2}\right)$ one has$\frac{1}{z_1^2\beta_1^{n+1}}\frac{1}{z_2^2\beta_2^{n+1}}\hat{\varphi} = \left(\frac{z_1\beta_1^n}{z_1^2\beta_1^{n+1}}\frac{2\beta_2^n}{z_2^2\beta_2^{n+1}}\right) = \left(\frac{1}{z_1\beta_1}\right)\left(\frac{1}{z_2\beta_2}\right)f$and so we get the function.

So we have the following

$$ZJ_2(f(x,y)) = \frac{\beta 1^n}{z1}\frac{\beta 2^n}{z2}\int_0^\infty\int_0^\infty \mathrm{f}(x,y)e^{-\frac{z1x}{\beta 1}-\frac{z2y}{\beta 2}}dxdy$$

And its Inverse

$$ZJ_2(f(x,y)) = \frac{z1\beta 1^n}{2\pi}\frac{z2\beta 2^n}{2\pi}\int_{c-\infty}^{c+\infty}\int_{c-\infty}^{c+\infty} f\left(\frac{1}{\beta_1\beta_2}\right)e^{z_1\beta_1 x+z_1\beta_2 y}dz_1\beta_1 dz_1\beta_2$$

Below are some examples of the application of the Double ZJ Transform

Volterra Integral Equation
Example 1

$$4y = \int_0^x\int_0^y f(x-\rho,y-\tau)f(\rho,\tau)d\rho d\tau$$

This is

$$4\left[\frac{\beta_1^{n+2}}{z_1^3}\right] = \left(\frac{\beta_1}{z_1}\right)\left(\frac{\beta_2}{z_2}\right)\hat{\varphi}^2$$

Which leaves us as

$$\hat{\varphi}^2 = 4\left[\frac{\beta_1^{n+1}}{z_1^2}\right]\left(\frac{z_2}{\beta_2}\right)$$

Taking the inverse we have $f\left(\frac{1}{\beta_1\beta_2}\right)$ and for$z_1\beta_1^n z_2\beta_2^n\hat{\varphi} = \frac{2}{\sqrt{z_1\beta_1}}$ y es $\frac{2}{\sqrt{\pi t}}$

Example 2

$$\frac{\partial^2 h}{\partial y^2} - \frac{\partial^2 h}{\partial x^2} + h(x,y) + \int_0^x\int_0^y w(x-\rho,y-\tau)h(\rho,\tau)d\rho d\tau = f(x,y)$$

$$h(x,0) = k1(x) \quad h(0,y) = g1(y)$$

$$\frac{\partial h(x,0)}{\partial y} = k2(x)\frac{\partial h(0,y)}{\partial x} = g2(y)$$

Let's take the simple case where h=f=0, k1=k2=g1=g2=0 the transformation is as follows

$$\frac{\partial^2 h}{\partial y^2} - \frac{\partial^2 h}{\partial x^2} - \int_0^x\int_0^y w(x-\rho,y-\tau)h(\rho,\tau)d\rho d\tau = 0$$

$$h(x,0) = 0 \quad h(0,y) = 0$$

$$\frac{\partial h(x,0)}{\partial y} = 0 \qquad \frac{\partial h(0,y)}{\partial x} = 0$$

$$\frac{\partial^2 h}{\partial y^2} = \left(\frac{z_2}{\beta_2}\right)^2\hat{\varphi}$$

$$\frac{\partial^2 h}{\partial x^2} = \left(\frac{z_1}{\beta_1}\right)^2\hat{\varphi}$$

$$\int_0^x\int_0^y (e^{x-\rho+y-\tau})h(\rho,\tau)d\rho d\tau = \left(\frac{z_1}{\beta_1^n}\right)\left(\frac{z_2}{\beta_2^n}\right)ZJ[e^{y+x}]\hat{\varphi}$$

$$ZJ[e^{y+x}] = \frac{\beta_2^2\beta_1^2}{(z_2(z_2-\beta_2))(z_1(z_1-\beta_1))}$$

The solution taking the inverse is left to us $f\left(\frac{1}{\beta_1\beta_2}\right)$ and for $z_1\beta_1^n z_2\beta_2^n$

$$\hat{\varphi}\left(1-\left(\left(\frac{z_1\beta_1}{z_2\beta_2}\right)-\left(\frac{z_2\beta_2}{z_1\beta_1}\right)\right)\frac{1}{(z_2\beta_2-1)(z_1\beta_1-1)}\right)=0$$

And this is how it remains for us $\hat{h}(e^{y+x}) = 0$ con $\hat{h} = 0$but it has another internal solution which ise^{y+x}

Example 3

$$\frac{\partial h}{\partial x}+\frac{\partial h}{\partial y} = -1+e^x+e^y+e^{x+y}+\int_0^x\int_0^y h(r,t)drdt$$

$$h(x,0)=\hat{\varphi}(u,0)=e^x \quad h(0,y)=\hat{\varphi}(0,v)=\ e^y$$

$$ZJ[e^{y+x}] = \frac{\beta_2^{n+1}\beta_1^{n+1}}{(z_2(z_2-\beta_2))(z_1(z_1-\beta_1))}$$

$$ZJ\left[\frac{\partial h}{\partial x}\right]=\frac{z_1}{\beta_1}\hat{\varphi}(u,v)-\frac{\beta_1^n}{z_1}\hat{\varphi}(0,v)$$

$$ZJ\left[\frac{\partial h}{\partial y}\right]=\frac{z_2}{\beta_2}\hat{\varphi}(u,v)-\frac{\beta_2^n}{z_2}\hat{\varphi}(u,0)$$

$$ZJ[e^x]=\frac{\beta_1^{n+1}}{(z_1(z_1-\beta_1))}$$

$$ZJ[e^y]=\frac{\beta_2^{n+1}}{(z_2(z_2-\beta_2))}$$

$$ZJ[-1]=\ -\left(\frac{\beta_2^{n+1}}{z_2^2}\right)\left(\frac{\beta_1^{n+1}}{z_1^2}\right)$$

$$ZJ_2\left[\int_0^x\int_0^y h(r,t)drdt\right]=\frac{\beta 1}{z1}\frac{\beta 2}{z2}\hat{\varphi} \quad o \quad \frac{\beta_1^{n+1}}{z_1^2}\frac{\beta_2^{n+1}}{z_2^2}\hat{\varphi}$$

To facilitate the calculations we can put n = 1, This is like

$$\left[\frac{1}{\frac{\beta_1}{z_1}}+\frac{1}{\frac{\beta_2}{z_2}}-\left(\frac{\beta_1}{z_1}\right)^2\left(\frac{\beta_2}{z_2}\right)^2\right]\hat{\varphi}$$

$$=\frac{\left(\frac{\beta_1}{z_1}\right)(\beta_2)^2}{z_2(z_2-\beta_2)}+\frac{\left(\frac{\beta_2}{z_2}\right)(\beta_1)^2}{z_1(z_1-\beta_1)}-\left(\frac{\beta_2}{z_2}\right)^2\left(\frac{\beta_1}{z_1}\right)^2+\frac{(\beta_1)^2}{z_1(z_1-\beta_1)}\left(\frac{\beta_2}{z_2}\right)^2+\frac{(\beta_2)^2}{z_2(z_2-\beta_2)}\left(\frac{\beta_1}{z_1}\right)^2$$
$$+\frac{(\beta_1)^2(\beta_2)^2}{z_1z_2(z_1-\beta_1)(z_2-\beta_2)}$$

Reducing and returning to the original variables gives us

$$\hat{\varphi}=\frac{\frac{\beta_1^{n+1}}{z_1^2}\frac{\beta_2^{n+1}}{z_2^2}}{\left(1-\frac{\beta_2}{z_2}\right)\left(1-\frac{\beta_1}{z_1}\right)}$$

The solution taking the inverse is left to us $f\left(\frac{1}{\beta_1\beta_2}\right)$ and for $z_1\beta_1^n z_2\beta_2^n$

$$\hat{h}=\frac{1}{(z_2\beta_2-1)((z_1\beta_1-1))}=e^{y+x}$$

Conclusions

We can conclude that the demonstration of the Double ZJ Transform follows the same criteria as the other Double Transforms such as Laplace and Elzaki, in addition to some examples when using the Double Transform to Volterra Integral Equation and Integro Differential Equations.

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