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The Double ZJ Transform and Its Application

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Abstract: The following article is a study and application about the double ZJ transform and its application to solve Partial Differential Equations.

Summary: The article is a study and application about the double ZJ transform its application for Partial Differential Equations.

Keywords: Double Laplace Transform, Double Integral Transform, PDE, Double ZJ Transform

1. Introduction

For engineering use, the answer and solution of various differential equations with initial and boundary conditions, ordinary and partial, is sought. Partial differential equations are solved in various ways using the method of separation of variables and reducing their solution to series, using integral transformations depending on the boundary conditions or not, common methods in engineering under a given geometry using numerical methods, decomposition methods, methods of change in the variables for their reduction etc... In the literature there are several articles with this reference, in this article that will be presented, the double ZJ transform is used to solve some partial differential equations and an application of the ordinary ZJ transform to improper integrals as an annex with a known technique.

Therefore, we start with the following definitions and properties to perform calculations

1.1 Definition of the Double ZJ Transform

Let f(x,y) with x, y > 0 be a function that can be expressed as a convergent infinite series, then its Double ZJ Transform is defined as long as integrals exist.

$$ZJ_2(f(x,y)) = \frac{\beta 1^n}{z1}\frac{\beta 2^n}{z2}\int_0^\infty\int_0^\infty \mathrm{f}(x,y)e^{-\frac{z1x}{\beta 1}-\frac{z2y}{\beta 2}}dxdy$$

The double inverse transform is defined as:

$$ZJ_2{}^{-1}(f(x,y)) = \frac{z1\beta 1^n}{2\pi}\frac{z2\beta 2^n}{2\pi}\int_{c-\infty}^{c+\infty}\int_{c-\infty}^{c+\infty} f\left(\frac{1}{\beta_1\beta_2}\right)e^{z_1\beta_1 x+z_1\beta_2 y}dz_1\beta_1 dz_1\beta_2$$

Exponential order function f(x,y) with a>0 b>0 in 0 <= x < +Inf, 0<= y < +Inf if there exist positive constants k such that

$$|f(x,y)| < k\, e^{-\frac{z1x}{\beta 1}-\frac{z2y}{\beta 2}}$$

2. Standard Properties of the Double ZJ Transform

A. Linearity property

If f(x,y) and g(x,y) with

$$ZJ_2(\alpha f(x,y)+\beta g(x,y)) = \alpha ZJ_2\big(f(x,y)\big)+\beta ZJ_2\big(g(x,y)\big) = \alpha T_1(u,v)+\beta T_2(u,v)$$

B. Change of Scale

If $ZJ_2\big(f(x,y)\big) = T(u,v)$ so $ZJ_2\big(f(ax,by)\big) = \frac{1}{ab}\frac{\beta_1^n}{z_1}\frac{\beta_2^n}{z_2}T(au,bv)$.

C. First Property of Change equal if it is –a or –b just replace

Si $ZJ_2\big(f(x,y)\big) = ZJ_2\big(e^{by+ax}f(x,y)\big) = \frac{\beta_1^{n+1}}{z_1}\frac{\beta_2^{n+1}}{z_2}T\left(\frac{1}{z_1-a\beta_1},\frac{1}{z_2-b\beta_2}\right)$.

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The demonstrations are immediate. Here T is the transformed function

Table 1. ZJ partial derivative transformation

$ZJ_2\left(\frac{\partial f(x,y)}{\partial x}\right)=\frac{z_1}{\beta_1}\hat{\varphi}(u,v)-\frac{\beta_1^n}{z_1}\hat{\varphi}(0,v)$ The Proof, using Integration by parts, the formula is demonstrated and also for the partial in y	$=\frac{\beta 1^n}{z1}\frac{\beta 2^n}{z2}\int_0^\infty\int_0^\infty \frac{\partial f(x,y)}{\partial x}e^{-\frac{z1x}{\beta 1}-\frac{z2y}{\beta 2}}dxdy=\frac{\beta 2^n}{z2}\int_0^\infty e^{-\frac{z2y}{\beta 2}}\left(\frac{\beta 1^n}{z1}\int_0^\infty \frac{\partial f(x,y)}{\partial x}e^{-\frac{z1x}{\beta 1}}dx\right)dy$ $=\frac{\beta 2^n}{z2}\int_0^\infty e^{-\frac{z2y}{\beta 2}}\left(\frac{z_1}{\beta_1}\hat{\varphi}(u,y)-\frac{\beta_1^n}{z_1}f(0,v)\right)dy$ $=\frac{z_1}{\beta_1}\left(\frac{\beta 2^n}{z2}\int_0^\infty e^{-\frac{z2y}{\beta 2}}\hat{\varphi}(u,y)dy\right)-\frac{\beta 1^n}{z1}\left(\frac{\beta 2^n}{z2}\int_0^\infty e^{-\frac{z1x}{\beta 1}}f(0,v)dy\right)$ $=\frac{z_1}{\beta_1}\hat{\varphi}(u,v)-\frac{\beta_1^n}{z_1}\hat{\varphi}(0,v)$ **(1)**

$ZJ_2\left(\frac{\partial f(x,y)}{\partial y}\right)=\frac{z_2}{\beta_2}\hat{\varphi}(u,v)-\frac{\beta_2^n}{z_2}\hat{\varphi}(u,0)$	**(2)**
$ZJ_2\left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)=\left(\frac{z_1}{\beta_1}\right)^2\hat{\varphi}(u,v)-\frac{\beta_1^n}{z_1}\frac{z_1}{\beta_1}\hat{\varphi}(0,v)-\frac{\beta_1^n}{z_1}\frac{\partial f(0,y)}{\partial x}$	**(3)**
$ZJ_2\left(\frac{\partial^2 f(x,y)}{\partial y^2}\right)=\left(\frac{z_2}{\beta_2}\right)^2\hat{\varphi}(u,v)-\frac{\beta_2^n}{z_2}\frac{z_2}{\beta_2}\hat{\varphi}(u,0)-\frac{\beta_2^n}{z_2}\frac{\partial f(x,0)}{\partial y}$	**(4)**
$ZJ_2\left(\frac{\partial^2 f(x,y)}{\partial x\partial y}\right)=\frac{z_1}{\beta_1}\frac{z_2}{\beta_2}\hat{\varphi}(u,v)-\frac{z_1}{\beta_1}\frac{\beta_2^n}{z_2}\hat{\varphi}(u,0)-\frac{\beta_1^n}{z_1}\frac{z_2}{\beta_2}\hat{\varphi}(0,0)+\frac{\beta_1^n}{z_1}\frac{\beta_2^n}{z_2}\hat{\varphi}(0,0)$	**(5)**
$ZJ_2(f(x))=\frac{\beta_2^n}{z_2}\frac{\beta_2}{z_2}\frac{\beta_1^n}{z_1}\hat{\varphi}(u)$	**(6)**
$ZJ_2(g(y))=\frac{\beta_1}{z_1}\frac{\beta_1^n}{z_1}\hat{\varphi}(v)$	**(7)**

Table 2. Double ZJ transform of some standard functions

Function	Transformation of the Function
1	$\frac{\beta_2^{n+1}}{z_2^2}\frac{\beta_1^{n+1}}{z_1^2}$
e^{ax+by}	$\frac{\beta_1^{n+1}\beta_2^{n+1}}{z_1(z_1-a\beta_1)z_2(z_2-b\beta_2)}$
$\cos(ax+by)$	$\frac{\beta_1^{n+1}\beta_2^{n+1}}{(z_1^2+a^2\beta_1^2)(z_2^2+b^2\beta_2^2)}-\frac{ab\beta_1^{n+2}\beta_2^{n+2}}{z_1z_2(z_1^2+a^2\beta_1^2)(z_2^2+b^2\beta_2^2)}$
$\mathrm{sen}\,(ax+by)$	$\frac{a\beta_1^{n+2}\beta_2^{n+1}}{z_1(z_1^2+a^2\beta_1^2)(z_2^2+b^2\beta_2^2)}+\frac{b\beta_1^{n+1}\beta_2^{n+2}}{z_2(z_1^2+a^2\beta_1^2)(z_2^2+b^2\beta_2^2)}$
$\cosh(ax+by)$	$\frac{\beta_1^{n+1}\beta_2^{n+1}}{(z_1^2-a^2\beta_1^2)(z_2^2-b^2\beta_2^2)}+\frac{ab\beta_1^{n+2}\beta_2^{n+2}}{z_1z_2(z_1^2-a^2\beta_1^2)(z_2^2-b^2\beta_2^2)}$
$\mathrm{senh}\,(ax+by)$	$\frac{a\beta_1^{n+2}\beta_2^{n+1}}{z_1(z_1^2-a^2\beta_1^2)(z_2^2-b^2\beta_2^2)}+\frac{b\beta_1^{n+1}\beta_2^{n+2}}{(z_1^2-a^2\beta_1^2)(z_2^2-b^2\beta_2^2)}$

Now we move on to the calculus examples

Examples

Non-homogeneous partial differential equations

$\frac{\partial^2 u(x,y)}{\partial x \partial y} = -c\, sen(cy)$	$u(x,0) = x \quad u(0,y) = 0 \quad u(0,0) = 0$

Applying the ZJ Transform, we have

$$ZJ_2\left(\frac{\partial^2 u(x,y)}{\partial x \partial y}\right) = \frac{z_1}{\beta_1}\frac{z_2}{\beta_2}\hat{\varphi}(u,v) - \frac{z_1}{\beta_1}\frac{\beta_2^n}{z_2}\hat{\varphi}(u,0) - \frac{\beta_1^n}{z_1}\frac{z_2}{\beta_2}(0) + \frac{\beta_1^n}{z_1}\frac{\beta_2^n}{z_2}(0)$$

$ZJ_2(u(x,0) = x) = \frac{\beta_1^{n+2}}{z_1^3}$	$ZJ_2(sen(cy)) = -\frac{c^2\beta_1^{n+1}\beta_2^{n+2}}{z_1^2 z_2(z_2^2 + c^2\beta_2^2)}$
$\frac{z_1}{\beta_1}\frac{z_2}{\beta_2}\hat{\varphi}(u,v) - \frac{z_1}{\beta_1}\frac{\beta_2^n}{z_2}\frac{\beta_1^{n+2}}{z_1^3} = -\frac{c^2\beta_1^{n+1}\beta_2^{n+2}}{z_1^2 z_2(z_2^2 + c^2\beta_2^2)}$	Solving and using the inverse transform we have $\hat{\varphi}(u,v) = \frac{\beta_1^{n+2}\beta_2^{n+1}}{z_1^3(z_2^2 + c^2\beta_2^2)}$ $ZJ_2^{-1}(f(x,y)) = x\cos(cy)$

Partial differential equations of order one

$\frac{\partial u(x,y)}{\partial x} = \frac{\partial u(x,y)}{\partial y}$	$u(x,0) = x \quad u(0,y) = y$
Taking the Transformation we have $ZJ_2\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{z_1}{\beta_1}\hat{\varphi}(u,v) - \frac{\beta_1^n}{z_1}\hat{\varphi}(0,v)$	$ZJ_2\left(\frac{\partial f(x,y)}{\partial y}\right) = \frac{z_2}{\beta_2}\hat{\varphi}(u,v) - \frac{\beta_2^n}{z_2}\hat{\varphi}(u,0)$
$ZJ_2(u(x,0)) = x = \frac{\beta_1^{n+2}}{z_1^3}$	$ZJ_2(u(0,y)) = y = \frac{\beta_2^{n+2}}{z_2^3}$

Thus

$$\left[\frac{z_1}{\beta_1}\hat{\varphi}(u,v) - \frac{\beta_1^n}{z_1}\frac{\beta_2^{n+2}}{z_2^3}\right] - \left[\frac{z_2}{\beta_2}\hat{\varphi}(u,v) - \frac{\beta_2^n}{z_2}\frac{\beta_1^{n+2}}{z_1^3}\right] = 0$$

The final equation is

$$\hat{\varphi}(u,v) = \frac{z_1^2}{\beta_1^{n+1}}\frac{z_2^3}{\beta_2^{n+2}} + \frac{z_2^2}{\beta_2^{n+1}}\frac{z_1^3}{\beta_1^{n+2}}$$

Solving and using the inverse transform we have

$$ZJ_2^{-1}(f(x,y)) = x + y$$

Non-homogeneous partial differential equations of order one

$\frac{\partial u(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} = e^y(1+x)$	$u(x,0) = x \quad u(0,y) = 0$

Taking the Transformation we have

$ZJ_2\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{z_1}{\beta_1}\hat{\varphi}(u,v) - \frac{\beta_1^n}{z_1}(0)$	$ZJ_2\left(\frac{\partial f(x,y)}{\partial y}\right) = \frac{z_2}{\beta_2}\hat{\varphi}(u,v) - \frac{\beta_2^n}{z_2}\frac{\beta_1^{n+2}}{z_1^3}$
$ZJ_2(e^y(1+x)) = \frac{\frac{\beta_2^{n+1}}{z_2^2}}{1 - \frac{\beta_2^n}{z_2}}\left(\frac{\beta_1^{n+1}}{z_1^2} + \frac{\beta_1^{n+2}}{z_1^3}\right)$	The result in g equation is as

	$\left[\frac{z_1}{\beta_1}+\frac{z_2}{\beta_2}\right]\hat{\varphi}(u,v)=\left(\frac{\frac{\beta_2^{n+1}}{z_2^2}}{1-\frac{\beta_2^n}{z_2}}\right)\left(\frac{\beta_1^{n+1}}{z_1^2}+\frac{\beta_1^{n+2}}{z_1^3}\right)+\frac{\beta_2^n}{z_2}\frac{\beta_1^{n+2}}{z_1^3}$

Solving and using the inverse transform we have

$$ZJ_2^{-1}(f(x,y))=xe^y$$

Partial differential equations of order two homogeneous

$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}-4u=0$	$u(x,0)=x \quad u(0,y)=y$ $\frac{\partial u(0,y)}{\partial x}=-2y$ $\frac{\partial u(x,0)}{\partial y}=e^{-2x}$

Taking the Transformation, we have

$ZJ_2(u(x,0))=x=\frac{\beta_1^{n+2}}{z_1^3}$	$ZJ_2(u(0,y))=y=\frac{\beta_2^{n+2}}{z_2^3}$
$ZJ_2\left(\frac{\partial u(0,y)}{\partial x}\right)=-2\frac{\beta_2^{n+2}}{z_2^3}$	$ZJ_2\left(\frac{\partial u(x,0)}{\partial y}\right)=\frac{\beta_1^{n+1}}{z_1(z_1+2\beta_1)}$

$$ZJ_2\left(\frac{\partial^2 f(x,y)}{\partial x^2}\right)=\left(\frac{z_1}{\beta_1}\right)^2\hat{\varphi}(u,v)-\frac{\beta_1^n}{z_1}\frac{z_1}{\beta_1}\hat{\varphi}(0,v)-\frac{\beta_1^n}{z_1}\frac{\partial f(0,y)}{\partial x}$$

The resulting equation is

$$\left[\left(\frac{z_1}{\beta_1}\right)^2\hat{\varphi}(u,v)-\frac{\beta_1^n}{\beta_1}\frac{\beta_2^{n+2}}{z_2^3}-\frac{\beta_1^n}{z_1}-2\frac{\beta_2^{n+2}}{z_2^3}\right]$$
$$+\left[\left(\frac{z_2}{\beta_2}\right)^2\hat{\varphi}(u,v)-\frac{\beta_2^n}{\beta_2}\frac{\beta_1^{n+2}}{z_1^3}-\frac{\beta_2^n}{z_2}\frac{\beta_1^{n+1}}{z_1(z_1+2\beta_1)}\right]-4\hat{\varphi}(u,v)=0$$

Solving and using the inverse transform we have

$$ZJ_2^{-1}(f(x,y))=ye^{-2x}$$

Conclusion

In this paper, we use the double ZJ transform to solve various types of partial differential equations and its standard properties of the transform. Using the results, different partial differential equations are solved by the double ZJ transform in a few steps, the only thing to be careful about is when solving the resulting algebraic equation and to facilitate the calculations you can set n = 1.

In addition, an annex of the use for Improper Integrals with a trigonometric structure in the numerator with a form similar to the double transform.

Appendix

In this section of the appendix to the article, the 1D ZJ transform will be used to solve some cases of improper integrals.

Definition 1.

The Transform is defined as:

$$\boldsymbol{ZJ\left[\left(f(t),\frac{Z}{\beta}\right)\right]=ZJ\left(\frac{Z}{\beta}\right)=\frac{\beta^n}{Z}\int_0^\infty e^{\frac{-Zt}{\beta}}f(t)dt} \qquad (1)$$

Inverse Transform

$$\boldsymbol{ZJ^{-1}[f(Z\beta),t)] = \frac{\beta^n Z}{2\pi i}\int_{-\infty}^{\infty} e^{Z\beta t} f\left(\frac{1}{\beta}\right) d\beta Z} \qquad (2)$$

Where z is a positive integer constant, β is the complex number and n is a positive integer.

Examples with Improper Integrals with the numerator as a trigonometric function.

Example 1

$$\int_{-\infty}^{\infty} \frac{\cos(x)\,dx}{1+x^2}$$

Taking the transform and putting a t in the cosine argument

$$\frac{\beta^n}{z}\int_0^{\infty}\left(\int_{-\infty}^{\infty}\frac{\cos(tx)\,dx}{1+x^2}\right)e^{\frac{-zt}{\beta}}dt$$

Now we have two functions, which are, and solving the Integral

$$\int_0^{\infty}\left(\frac{1}{1+x^2}\right)\left(\frac{\beta^{n+1}}{z^2+x^2\beta^2}\right)dx = \frac{\pi\beta^{n+1}}{2z(z+\beta)}$$

Now taking the Inverse of the Transformation we have the solution and remember that it is $-\infty < x < \infty$ thus with t = 1

$$\int_{-\infty}^{\infty}\frac{\cos(x)\,dx}{1+x^2} = \frac{\pi}{e}$$

Example 2

$$\int_0^{\infty}\frac{\text{sen}(x)\,dx}{x}$$

Taking the transform and putting a t in the sine argument

$$\frac{\beta^n}{z}\int_0^{\infty}\left(\int_0^{\infty}\frac{\text{sen}(\text{tx})dx}{x}\right)e^{\frac{-zt}{\beta}}dt$$

Now we have two functions which are and solving by the Inverse Transform to the Sine transformation we have the following, now as it is only the interval 0 < x <∞, it is only π the integral thus with the sine it diverges, so taking a branch in its complex exponential terms as in Complex Analysis we have the result or completely solving the Integral gives us the demonstrated result.

$$\int_0^{\infty}\left(\frac{1}{x}\right)\left(\frac{x\beta^{n+2}}{z(z^2+x^2\beta^2)}\right)dx = \int_0^{\infty}(sen(x))dx = \pi\int_0^{\infty}\left(\frac{e^{ix}}{2i}\right)dx = \frac{\pi}{2}$$

This technique can also be corroborated by the complex analysis residue method.

Example 3

$$\int_0^{\infty}\frac{x\,\text{sen}\,3(x)\,dx}{(4+x^2)^2}$$

Taking the transform and putting a t in the sine argument

$$\frac{\beta^n}{z}\int_0^\infty\left(\int_0^\infty \frac{x\,\mathrm{sen}\,3(x)\,dx}{(4+x^2)^2}\right)e^{\frac{-zt}{\beta}}dt$$

$$sen(3tx)=\frac{3x\beta^{n+2}}{z(z^2+9x^2\beta^2)}$$

Now the Integral is left to us

$$\int_0^\infty \frac{3x^2dx}{(4+x^2)^2}*\frac{\beta^{n+2}}{z(z^2+9x^2\beta^2)}$$

Solving the Integral with respect to dx, taking the Invers e$f\left(\frac{1}{\beta}\right)$ and with $Z\beta^n$ and t = 1 you have the solution

$$=\frac{3\pi\,e^{-6}}{8}$$

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