International Journal of Latest Research in Engineering and Technology (IJLRET)
ISSN: 2454-5031
www.ijlret.com || Volume 11 - Issue 07 || July 2025 || PP. 17-26

A Solution to Nonlinear Ordinary Differential Equations using the Adomian Decomposition Method and the ZJ Transform

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Abstract: In this article, we will explore the Adomian Decomposition Method, its theoretical foundation, and the computation of Adomian polynomials. Furthermore, we will investigate the application of the ZJ transform in conjunction with MDA for the solution of first- and second-order nonlinear ODEs. Through illustrative examples, we will analyze the effectiveness and potential of this hybrid approach for obtaining approximate analytical solutions to relevant nonlinear problems.

Keywords: ZJ transform, Laplace transform, Adomian decomposition method, Adomianpolynomials, Nonlinear differential equations, Series

Introduction:

Nonlinear ordinary differential equations (ODEs) model a wide range of complex phenomena in diverse scientific and engineering disciplines, from fluid dynamics and heat transfer to biology and economics. However, the intrinsically nonlinear nature of these equations often makes exact analytical solutions difficult to obtain. Consequently, the development and application of robust and efficient approximate numerical and analytical methods has become an active and crucial area of research.

Among the powerful analytical techniques for addressing nonlinear ODEs, the Adomian Decomposition Method (ADM) has emerged as a versatile and effective tool. Introduced by George Adomian in the 1980s, ADM offers a methodology for obtaining convergent series solutions without requiring linearization, discretization, or the introduction of small perturbation parameters, features that often limit the applicability of other traditional methods. The cornerstone of ADM lies in the decomposition of the nonlinear operator present in the equation into a series of specific polynomials, known as Adomian polynomials, which depend on the series components of the solution.

In parallel, the use of integral transforms, such as the Laplace transform, has proven invaluable in simplifying and solving differential equations, especially those with linear terms and well-defined initial conditions. The Laplace transform converts a differential equation into an algebraic equation in the transform domain, which is often easier to solve. Subsequently, applying the inverse transform allows the solution to be obtained in the original domain.

The strategic combination of the Adomian Decomposition Method with integral transform techniques has led to promising hybrid approaches for the treatment of nonlinear ODEs. By applying an integral transform to the original differential equation, the linear terms can be simplified, while the MDA handles the nonlinear part by generating the corresponding Adomian polynomials. The solution in the transform domain is then obtained in series form, and applying the inverse transform provides the solution in the original domain.

This combined approach can offer several advantages, including the potential acceleration of the convergence of the solution series, simplification of the handling of initial conditions, and the obtaining of approximate analytical solutions with a high degree of accuracy.

Overview of the Combined Method (Adomian-ZJ) for Second-Order Nonlinear ODEs

Let us consider a second-order nonlinear ODE of the general form:

$$L\big(u(x,t)\big) + R\big(u(x,t)\big) + N\big(u(x,t)\big) = h(x,t)$$

We have:

- $L(u(x,t))=au''(x,t)+bu'(x,t)$ represents the highest order linear operator (with $a, b \geq 0$).
- $R(u(x, t))$ represents the remaining lower-order linear operator.
- $N(u(x, t))$ represents the nonlinear operator.
- $h(x, t)$ is the non-homogeneous function (source term).

Along with the initial conditions:

$U(0)=c0, u'(0)=c1$ o $u(0) = f(x) u^{'}(0) = g(x)$

Step 1: Applying the ZJ Transform

We apply the ZJ transform, denoted by ZJ {·}, to both sides of the ODE:

ZJ{L(u(x, t))}+ZJ{R(u(x, t))}+ZJ{N(u(x, t))}=ZJ{h(x, t)}

Using the properties of the ZJ transform for first and second order derivatives with constant coefficients:

$ZJ\{u'(x,t)\}=\frac{z}{\beta}\hat{\varphi}-y(x,0)\frac{\beta^n}{z}=\frac{z}{\beta}\hat{\varphi}-c0\frac{\beta^n}{z}$

$ZJ\{u''(x,t)\}=\frac{z^2}{\beta^2}\hat{\varphi}-\ y(x,0)\frac{\beta^n}{\beta}-y'(x,0)\frac{\beta^n}{z}=\frac{z^2}{\beta^2}\hat{\varphi}-c0\frac{\beta^n}{\beta}-c1\frac{\beta^n}{z}$

Applying the transform to linear operators:

$ZJ\{L(u(x,t))\}=a(\frac{z^2}{\beta^2}\hat{\varphi}-c0\frac{\beta^n}{\beta}-c1\frac{\beta^n}{z})+b(\frac{z}{\beta}\hat{\varphi}-c0\frac{\beta^n}{z})$

$ZJ\{R(u(x,t))\}=L\{r_1u'(x,t)+r_0u(x,t)\}=r_1(\frac{z}{\beta}\hat{\varphi}-c0\frac{\beta^n}{z})+r_0ZJ(\hat{\varphi})$

(Here, we assume that R(u(t)) is a first or zero-order linear operator with constant coefficients r_1 andr_0 to simplify the notation.

The transform of the nonlinear part remains in its general form for now:

ZJ{N(u(x, t))}=ZJ{N(u)}

And the transform of the source term isZJ(H)=ZJ{h(x, t)}.

Substituting these expressions into the transformed equation:

$a(\frac{z^2}{\beta^2}\hat{\varphi}-c0\frac{\beta^n}{\beta}-c1\frac{\beta^n}{z})+b(\frac{z}{\beta}\hat{\varphi}-c0\frac{\beta^n}{z})+r_1(\frac{z}{\beta}\hat{\varphi}-c0\frac{\beta^n}{z})+r_0ZJ(\hat{\varphi})+ZJ\{N(u)\}=ZJ(H)$

Step 2: Algebraic Solution in the Domain of ZJ

We regroup the equation to solve algebraically for ZJ ($\hat{\varphi}$), isolating the terms that depend on ZJ ($\hat{\varphi}$) in a broken-down way:

$\left(a\frac{z^2}{\beta^2}+b\frac{z}{\beta}+r_1\frac{z}{\beta}+r_0\right)ZJ(\hat{\varphi})=ZJ(H)+a\left(c0\frac{\beta^n}{\beta}+c1\frac{\beta^n}{z}\right)+bc0\frac{\beta^n}{z}+r_1c0\frac{\beta^n}{z}-ZJ\{N(u)\}$

And with the initial conditions not being constant, we now have:

$\frac{z^2}{\beta^2}\hat{\varphi}-f(x)\frac{\beta^n}{\beta}-g(x)\frac{\beta^n}{z}+ZJ\{R(u(t))\}+ZJ\{N(u(t))\}=ZJ\{h(t)\}$

We regroup the equation to solve algebraically for ZJ($\hat{\varphi}$):

$$\hat{\varphi}=\frac{f(x)\frac{\beta^n}{\beta}+g(x)\frac{\beta^n}{z}}{\frac{z^2}{\beta^2}}+\frac{ZJ\{h(t)\}}{\frac{z^2}{\beta^2}}-\left(\frac{ZJ\{R(u(t))\}\ +\ ZJ\{N(u(t))\}}{\frac{z^2}{\beta^2}}\right)$$

Step 3: Application of the Adomian Decomposition Method

Now, we apply the MDA to solve the algebraic equation in the domain ZJ. We assume that the solution u(t) can be expressed as an infinite series. Using the superposition principle, the solution can be represented as an infinite series. $\sum_{n=0}^{\infty}\hat{\varphi}_n(x,t)$

If we observe the nonlinear operator Nu(x, t) from our equation, we decompose it as a series of Adomianpolynomials as Nu(x, t) = $\sum_{n=0}^{\infty}A_n(x,t)$ where An are Adomian polynomials of u_n and it can be determinedby the relation

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\sum_{i=0}^{\infty}\lambda^n u_i\right]$$

Using our previous equations, we can recast

$$ZJ\left(\sum_{n=0}^{\infty}\hat{\varphi}_n(x,t)\right)=\frac{f(x)\frac{\beta^n}{\beta}+g(x)\frac{\beta^n}{z}}{\frac{z^2}{\beta^2}}+\frac{ZJ\{h(t)\}}{\frac{z^2}{\beta^2}}-\frac{ZJ\{R(u(t))\}}{\frac{z^2}{\beta^2}}-\frac{ZJ\{\sum_{n=0}^{\infty}A_n(x,t)\}}{\frac{z^2}{\beta^2}}$$

When we compare both sides of the equations in terms of $\hat{\varphi}_1$, $\hat{\varphi}_0$, $\hat{\varphi}_{n+1}$, we have

$$\sum_{n=0}^{\infty}\hat{\varphi}_0(x,t)=\frac{f(x)\frac{\beta^n}{\beta}+g(x)\frac{\beta^n}{z}}{\frac{z^2}{\beta^2}}+\frac{ZJ\{h(t)\}}{\frac{z^2}{\beta^2}}$$

$$\sum_{n=0}^{\infty}\hat{\varphi}_1(x,t)=-\frac{ZJ\{R(\hat{\varphi}_0(x,t))\}}{\frac{z^2}{\beta^2}}-\frac{ZJ\{\sum_{n=0}^{\infty}A_0(x,t)\}}{\frac{z^2}{\beta^2}}$$

$$\sum_{n=0}^{\infty} \hat{\varphi}_2(x,t) = -\frac{\mathrm{ZJ}\{\mathrm{R}(\hat{\varphi}_1(x,t))\}}{\frac{z^2}{\beta^2}} - \frac{\mathrm{ZJ}\{\sum_{n=0}^{\infty} A_1(x,t)\}}{\frac{z^2}{\beta^2}}$$

The preceding terms, depending on how far of the approximation can be determined by the given recursiverelation con $n \geq 1$:

$$\sum_{n=0}^{\infty} \hat{\varphi}_{n+1}(x,t) = -\frac{\mathrm{ZJ}\{\mathrm{R}(\hat{\varphi}_n(x,t))\}}{\frac{z^2}{\beta^2}} - \frac{\mathrm{ZJ}\{\sum_{n=0}^{\infty} A_n(x,t)\}}{\frac{z^2}{\beta^2}}$$

The accuracy of the approximation increases with the number of terms included in the sum, now. If we apply the inverse ZJ transform to the above equation, we get

$$u_{n+1}(x,t) = -ZJ^{-1}\left(\frac{\mathrm{ZJ}\{\mathrm{R}(\hat{\varphi}_n(x,t))\}}{\frac{z^2}{\beta^2}} + \frac{\mathrm{ZJ}\{\sum_{n=0}^{\infty} A_n(x,t)\}}{\frac{z^2}{\beta^2}}\right)$$

Example 1

Let the Nonlinear Second Order Ordinary Differential Equation be

$$\boldsymbol{y'' + e^{y} = 0}$$

With $\boldsymbol{y(0) = 1 y'(0) = 0}$

By the ZJ Transformed, transforming the E.D.O.

$$ZJ\,[y''] + ZJ\,[e^{y}] = 0$$

$$y'' = -\frac{\beta''}{z} y'(x,u) - \frac{\beta''}{\beta} y(x,0) + \frac{z^2}{\beta^2}\hat{\varphi} \text{ with } I.C.$$

$$y'' = -\frac{\beta''}{z} + \frac{z^2}{\beta^2}\hat{\varphi} \quad thus \ -\frac{\beta''}{z} + \frac{z^2}{\beta^2}\hat{\varphi} = -ZJ\,[e^{y}]$$

$$\hat{\varphi} = \frac{\beta^2}{z^2}\left(\frac{\beta''}{z}\right) - \frac{\beta^2}{z^2} ZJ\,\{e^{y}\} \ \ Now\ the\ reverse$$

$$y(t) = \frac{\beta^{n+2}}{z^3} - \frac{\beta^2}{z^2} ZJ\,\{e^{y}\} = ZJ^{-1}\left(\frac{\beta^{n+2}}{z^3}\right) - ZJ^{-1}\left[\frac{\beta^2}{z^2} ZJ\,\{e^{y}\}\right]$$

$$y(t) = \left(\frac{1}{z^3\beta^{n+2}}\right) * z\beta^n = \frac{1}{z^2\beta^2} = t \ \ thus \ \ t = y_0$$

$$y(t) = t - ZJ^{-1}\left(\frac{\beta^2}{z^2} ZJ[e^{y_0}]\right) \ \ thus\ ZJ(e^{t})$$

$$and\ this\ is\ \frac{\beta^{n+1}}{z(z-\beta)} \therefore\ therefore\ by\ the\ Adomian\ method$$

$$y(t) = t - ZJ^{-1}\left(\frac{\beta^2}{z^2}\left(\frac{\beta^{n+1}}{z(z-\beta)}\right)\right) = t - ZJ^{-1}\left(\frac{\beta^{n+3}}{z^3(z-\beta)}\right)$$

Now taking inverse

$$\approx \left(\frac{1}{\beta^{n+3} z^3\left(z - \frac{1}{\beta}\right)}\right) = \frac{1}{z^3\beta^{n+3}\frac{(z\beta-1)}{\beta}} = \frac{1}{z^3\beta^{n+2}(z\beta-1)} * z\beta^n = \frac{1}{z^2\beta^2(z\beta-1)}$$

$$= with\ y_1 = t - e^{t} + 1$$

Now n=1

The Adomian Series is

Now the Adomian polynomials

$$A_1 = y_1 e^{y^0}$$

$$A_2 = (y_2 + \frac{{y_2}^2}{2})e^{y_0}$$

$$A_3 = (y_3 + y_1 y_2 + \frac{y^2{}_1}{2})e^{y_0}$$

$$y_1 = t - e^{t} + 1$$

$$y_0 = t$$

$$y_{n+1} = -ZJ^{-1}\left(\frac{\beta^2}{z^2} ZJ[e^{y}]\right)$$

$$y_2 = -ZJ^{-1}\left(\frac{\beta^2}{z^2} ZJ[y_1 e^{y_0}]\right)$$

Now replacing

$$y_2 = -ZJ^{-1}\left(\frac{\beta^2}{z^2} ZJ[t - e^t + 1]e^t\right) thus$$

$$ZJ[te^t - e^{2t} + e^t] The transform is t^n e^{kt} = \frac{n!\,\beta^n}{z\left(k - \frac{z}{\beta}\right)^{n+1}}$$

$$te^t = \frac{\beta^n}{z\left(1 - \frac{z}{\beta}\right)^2}$$

And thus solving each term by its transform and its inverse

$$y_2 = \frac{e^{2t}}{4} - te^t + e^t - \frac{t}{2} - \frac{5}{4}$$

$$y = t - (e^t - t - 1) + \left(\frac{e^{2t}}{4} - te^t + e^t - \frac{t}{2} - \frac{5}{4}\right)$$

Simplifying we have the solution

$$\boldsymbol{y(t) \approx \frac{3t}{2} - \frac{1}{4} + \frac{e^{2t}}{4} - te^t}$$

Example 2
Let the Nonlinear Second Order Ordinary Differential Equation be

$$\boldsymbol{y'' + y^2 = 0}$$

With $y(0) = 1 y'(0) = 0$
By the ZJ Transformed, transforming the E.D.O.

$$ZJ[Y''] + ZJ[y^2] = 0 \quad \rightarrow ZJ[y''] = -ZJ[y^2]$$

$$y'' = -\frac{\beta^n}{Z} y'^{(x,0)} - \frac{\beta^n}{\beta} y(x,0) + \frac{z^2}{\beta^2} y(x,0) + \frac{z^2}{\beta^2}\hat{\varphi} with the I.C$$

$$y'' = 0 - \frac{\beta^n}{\beta} + \frac{z^2}{\beta^2}\hat{\varphi} thus$$

$$\frac{z^2}{\beta^2}\hat{\varphi} = \frac{\beta^n}{\beta} - ZJ[y^2]$$

$$\hat{\varphi} = \frac{\beta^{n+2}}{z^2\beta} - \frac{\beta^2}{z^2} ZJ[y^2]$$

$$\hat{\varphi} = \frac{\beta^{n+1}}{z^2} - \frac{\beta^2}{z^2} ZJ[y^2] Taking the inverse ZJ$$

$$y(t) = ZJ^{-1}\left[\frac{\beta^{n+1}}{z^2}\right] - ZJ^{-1}\left[\frac{\beta^2}{z^2} ZJ[y^2]\right]$$

$$y(t) = \left(\frac{1}{z^2\beta^{n+1}}\right) * Z\beta^n = \frac{1}{z\beta} = 1 \ thus$$

$$y(t) = 1 - ZJ^{-1}\left[\frac{\beta^2}{z^2} ZJ[y^2]\right]$$

ByAdomian

$$\boldsymbol{y_0 = 1 \text{ we have } y_0 = 1 = y_0^2}$$

$$\boldsymbol{y_{n+1} = -ZJ^{-1}\left[\frac{\beta^2}{z^2} ZJ[y^2]\right] \therefore for n = 0}$$

$$y_{1=} - ZJ^{-1}\left[\frac{\beta^2}{z^2} ZJ[y_0^2]\right] = - - ZJ^{-1}\left[\frac{\beta^2}{z^2} ZJ[1]\right]$$

$$= -ZJ^{-1}\left[\frac{\beta^2}{z^2}\left(\frac{\beta^{n+1}}{z^2}\right)\right] = -ZJ^{-1}\left[\frac{\beta^{n+1}}{z^4}\right]$$

$$y_1 = \left(\frac{1}{Z^4\beta^{n+3}}\right) * Z\beta^n = \left(\frac{1}{Z^3\beta^3}\right) = -\frac{t^2}{2}$$

$$Now A_1 = 2y_0y_1 Adomianpolynomials A_1 = 2(1)\left(-\frac{t^2}{2}\right) = -t^2\ thus$$

Now with n=1

$$y_2 = -ZJ^{-1}\left[\frac{\beta^2}{z^2}ZJ[-A_1]\right] = -ZJ^{-1}\left[\frac{\beta^2}{z^2}ZJ[-t^2]\right]$$

$$= -ZJ^{-1}\left[\frac{\beta^2}{Z^2}\left(-\frac{2\beta^{n+3}}{Z^4}\right)\right] = -ZJ^{-1}\left[-\frac{2\beta^{n+5}}{Z^6}\right]$$

$$\approx 2\left(\frac{\beta^{n+5}}{z^6}\right) = \frac{2}{z^6\beta^{n+5}} * z\beta^n = \frac{2}{z^5\beta^5} = \frac{t^4}{12}$$

And so the solution is $\boldsymbol{y(t) = 1 - \frac{t^2}{2} + \frac{t^4}{12} - \frac{t^6}{12} + \cdots}$

Example 3
Let the Nonlinear First Order Ordinary Differential Equation be

$$\mathbf{y' = y^3 y(0) = 1}$$

Porla Transformada ZJ, transformando la E.D.O

$$ZJ[y'] = ZJ[y^3] así \rightarrow -y(x,0)\frac{\beta^n}{z} + \frac{z}{\beta}\hat{\varphi}$$

$$\rightarrow -\frac{\beta^n}{z^2} + \frac{z}{\beta}ZJ[y^3] \rightarrow \hat{\varphi} = \frac{\beta}{z}\left[\frac{\beta^n}{z} + ZJ[y^3]\right]$$

$$\hat{\varphi} = \frac{\beta^{n+1}}{z^2} + \frac{\beta}{Z}ZJ[y^3]$$

$$ZJ^{-1}\left[\frac{\beta^{n+1}}{z^2}\right] = \frac{1}{z^2\beta^{n+1}} * z\beta^n = \frac{1}{z\beta} = 1$$

$$y = 1 + ZJ^{-1}\left[\frac{\beta}{z}ZJ[y^3]\right]$$

The Adomian Series

$$y_0 = 1$$

$$y_{n+1} = ZJ^{-1}\left[\frac{\beta}{z}ZJ[y^3]\right]$$

$$ByAdomianthisisy = \sum_{n=0}^{\infty} y_n(x)$$

$$\sum_{n=0}^{\infty} y_n(x) = 1 + ZJ^{-1}\left[\frac{\beta}{z}ZJ\left[\sum_{n=0}^{\infty} y_n(x)\right]\right]$$

With $y^3 = \sum_{n=0}^{\infty} An(x) thusis = y_0^3\ withy_0^3 = (1)^3$

Now with n=0

$$y_1 = ZJ^{-1}\left[\frac{\beta}{z}ZJ[y_0^3]\right] = ZJ^{-1}\left[\frac{\beta}{z}ZJ[1]\right]$$

$$= ZJ^{-1}\left[\frac{\beta}{z}\left[\frac{\beta^{n+1}}{z^2}\right]\right] = ZJ^{-1}\left[\frac{\beta^{n+2}}{z^3}\right]$$

Now the reverse

$$\approx \frac{1}{z^3\beta^{n+2}} * z\beta^n = \frac{1}{z^2\beta^2} = tandnowy_1 = t$$

Now with n=1

$$y_2 = ZJ^{-1}\left[\frac{\beta}{z}ZJ\left[\sum_{n=0}^{\infty}A_1\right]\right] withtheAdomianpolynomialA_1 = 3y_0^3y_1$$

And

$$y_2 = ZJ^{-1}\left[\frac{\beta}{z}ZJ\left[\sum_{n=0}^{\infty}[3y_0^3y_1]\right]\right]$$

$$y_2 = ZJ^{-1}\left[\frac{\beta}{z}ZJ[3(1)^3(t)]\right] = ZJ^{-1}\left[\frac{\beta}{z}ZJ[3t]\right]$$

$$= \frac{\beta}{z}\left[3\left(\frac{\beta^{n+2}}{z^3}\right)\right] = 3\frac{\beta^{n+3}}{z^4}$$

$$\approx \frac{3}{z^4\beta^{n+3}} = \frac{3 * z\beta^n}{z^4\beta^{n+3}} = \frac{3}{z^3\beta^3}$$

$$\approx \frac{3t^2}{2} withy_2 = \frac{3t^2}{2} thesolutionis$$

$$\boldsymbol{y = y_0 + y_1 + \frac{1}{2}\dots}$$

$$\boldsymbol{y = 1 + t + \frac{3t^2}{2} + \cdots}$$

Example 4
Let the Nonlinear First Order Ordinary Differential Equation be

$$\boldsymbol{y' + y^2 = ty(0) = 0}$$

So again applying the Transform $ZJ[y'] + ZJ[y^2] = ZJ[t]$

$$ZJ[y'] = -y(x,0)\frac{\beta^n}{z} + \frac{z}{\beta}\hat{\varphi}\widehat{y(0)} = 0$$

$$ZJ[y'] = \tfrac{z}{\beta}\hat{\varphi} so \tfrac{z}{\beta}\hat{\varphi} + ZJ[y^2] = ZJ(t)$$

$$\frac{z}{\beta}\hat{\varphi} + \sum_{n=0}^{\infty}ZJ[A_n(t)] = ZJ[t] = \frac{\beta^{n+2}}{z^3}así$$

$$\hat{\varphi} = \frac{\beta^{n+3}}{z^4} - \frac{\beta}{z}\sum_{n=0}^{\infty}ZJ[A_n(t)] \textbf{ Now using Adomian}$$

$$\sum_{n=0}^{\infty}\hat{\varphi}_n = \frac{\beta^{n+3}}{z^4} - \frac{\beta}{z}\sum_{n=0}^{\infty}ZJ[An(t)]$$

The Adomian Series

$$\hat{\varphi}_0 = \frac{\beta^{n+3}}{z^4}$$

$$\hat{\varphi}_{n+1} = -\frac{\beta}{z}ZJ\left[A_n(y_0, y_1, \dots, y_{n)}\right]n \geq 0$$

Now applying the inverse with n= 0

$$ZJ^{-1}[\hat{\varphi}_1] = ZJ^{-1}\left[\frac{\beta^{n+3}}{z^4}\right]asiy_0 = \frac{t^2}{2}$$

$Fory_1(t)$

$$A_0 = y_0^2 = \left(\frac{t^2}{2}\right)^2 = \frac{t^4}{4}NowZJ[A_0]$$

$$ZJ\left[\frac{t^4}{4}\right] = 6\frac{\beta^{n+4}}{z^6}Nowfind\varphi_1$$

$$\hat{\varphi}_1 = -\frac{\beta}{z}ZJ[A_0] = -\frac{\beta}{z}\left(6\frac{\beta^{n+4}}{z^6}\right) = -6\frac{\beta^{n+5}}{z^7}$$

Now the reverse

$$ZJ^{-1}\left[-6\frac{\beta^{n+5}}{z^7}\right] = -\frac{t^5}{20} sothesolutionis$$

$$\boldsymbol{y = \frac{t^2}{2} - \frac{t^5}{20} + \cdots}$$

Example 5

Let the Nonlinear First Order Ordinary Differential Equation be

$$\boldsymbol{y' = 2ky - 3ky^2 + ky^3 y(0) = \frac{1}{2}}$$

So again applying the Transform

$$ZJ[y'] = ZJ[2Ky] - ZJ[3Ky^2] + ZJ[Ky^3]$$

$$-y(x,0)\frac{\beta^n}{z} + \frac{z}{\beta}\hat{\varphi} = 2KZJ[y] - 3KZJ[y^2] + KZJ[y^3]$$

$$-0.5\frac{\beta^n}{z} + \frac{z}{\beta}\hat{\varphi} = 2KZJ[y] - 3KZJ[y^2] + KZJ[y^3]$$

$$\hat{\varphi} = 0.5\frac{\beta^{n+1}}{z^2} + 2\frac{\beta}{z}KZJ[y] - 3\frac{\beta}{z}KZJ[y^2] + \frac{\beta}{z}KZJ[y^3]$$

Now the reverse

$$y(t) = ZJ^{-1}\left[0.5\frac{\beta^{n+1}}{z^2}\right] + ZJ^{-1}\left[2\frac{\beta K}{z}ZJ[y]\right] - ZJ^{-1}\left[3\frac{\beta K}{z}Z[y^2]\right] + ZJ^{-1}\left[\frac{\beta}{z}KZJ[y^3]\right]$$

Now

$$ZJ^{-1}\left[0.5\frac{\beta^{n+1}}{z^2}\right] = 0.5\, ZJ^{-1}\left[\frac{\beta^{n+1}}{z^2}\right] = 0.5\left(\frac{1}{z^2\beta^{n+1}}\right) * zB^n$$

$$= 0.5\left(\frac{1}{z\beta}\right) = 0.5\ \ thisisy_0 = 0.5$$

Thus by the Adomian Series

$$y_0 = 0.5$$

$$y_{n+1} = ZJ^{-1}\left[2\frac{\beta K}{z}ZJ[y_0]\right] - ZJ^{-1}\left[3\frac{\beta K}{z}ZJ[y_0]^2\right] + ZJ^{-1}\left[\frac{\beta K}{z}Z[y_0]^3\right]$$

Now withn=0

$$y_1 = ZJ^{-1}\left[2\frac{\beta K}{z}ZJ[0.5]\right] = ZJ^{-1}\left[\frac{\beta K}{z}ZJ[1]\right]$$

$$y_1 = ZJ^{-1}\left[3\frac{\beta K}{z}ZJ[0.5]^2\right] = ZJ^{-1}\left[0.75\frac{\beta K}{z}ZJ[1]\right]$$

$$y_1 = ZJ^{-1}\left[\frac{\beta K}{z}ZJ[0.5]^3\right] = ZJ^{-1}\left[0.125\frac{\beta K}{z}ZJ[1]\right]$$

Now with each one

$$= ZJ^{-1}\left[\frac{\beta K}{z}\left(\frac{\beta^{n+1}}{z^2}\right)\right] = \frac{\beta^{n+2}k}{z^3} = ZJ^{-1}\left[\frac{K\beta^{n+2}}{z^2}\right]$$

$$= K\left[\frac{1}{z^3\beta^{n+2}}\right] = k\left[\frac{z\beta^n}{z^3\beta^{n+2}}\right] = K\left[\frac{1}{z^2\beta^2}\right] = Kt$$

$$= -ZJ^{-1}\left[0.75\frac{K\beta}{z}\left(\frac{\beta^{n+1}}{z^2}\right)\right] = -ZJ^{-1}\left[0.75\frac{K\beta^{n+2}}{z^3}\right] = 0.75K\left[\frac{\beta^{n+2}}{z^3}\right]$$

Now the reverse

$$= -\frac{0.75k}{z^3\beta^{n+2}} * z\beta^n = -\frac{0.75k}{z^2\beta^2} = -0.75kt$$

$$= ZJ^{-1}\left[0.125\frac{\beta K}{z}\left(\frac{\beta^{n+1}}{z^2}\right)\right] \approx 0.125kt$$

$$y_1 = kt - 0.75kt + 0.125kt = (k - 0.75k + 0.125k)t$$

Thus $y_1 = 0.375kt$ with the y_0 we have$y = 0.5 + 0.375kt$

Now n=1, with the Adomian polynomials

$$y_1^2 = A_1 = 2y_0y_1y\beta_1 = 3y_0^2y_1 = y_1^3$$

$$A_1 = 2(0.5)(0.375kt) = 0.375kt$$

$$B_1 = 3(0.5)^2(0.375kt) = 0.28125kt$$

$$y_2 = ZJ^{-1}\left[2\frac{k\beta}{z}ZJ[y_1]\right] = ZJ^{-1}\left[2\frac{k\beta}{z}ZJ[0.375kt]\right]$$

$$y_2 = ZJ^{-1}\left[3\frac{k\beta}{z}ZJ[y_1^2]\right] = ZJ^{-1}\left[3\frac{k\beta}{z}ZJ[A_1]\right]$$

$$y_2 = ZJ^{-1}\left[\frac{k\beta}{z}ZJ[y_1^3]\right] = ZJ^{-1}\left[2\frac{k\beta}{z}ZJ[B_1]\right]$$

We proceed

$$ZJ^{-1}\left[2\frac{k\beta}{z}(0.375k)ZJ[t]\right] = ZJ^{-1}\left[0.75\,\frac{k^2}{z}\beta ZJ[t]\right]$$

$$ZJ^{-1}\left[0.75\frac{k^2}{z}\beta\left(\frac{\beta^{n+2}}{z^3}\right)\right] = ZJ^{-1}\left[0.75\frac{k^2\beta^{n+3}}{z^4}ZJ(t)\right]$$

$$= 0.75k^2\left(\frac{1}{z^4\beta^{n+3}}\right) * z\beta^n = 0.75k^2\left(\frac{1}{z^3\beta^3}\right) =$$

$$= 0.75k^2\frac{t^2}{2} = 0.375k^2t^2$$

The Second Term

$$= -ZJ^{-1}\left[\frac{3\beta k}{z}ZJ[0.375kt]\right] = -ZJ^{-1}\left[1.125\frac{k^2}{z}\beta ZJ[t]\right]$$

$$= -ZJ^{-1}\left[1.125\frac{k^2\beta^{n+3}}{z^4}\right] \to= -1.125k^2\left(\frac{1}{z^4\beta^{n+3}}\right) * z\beta^n = \frac{-1.125\,k^2}{z^3\beta^3}$$

$$= -1.125k^2\frac{t^2}{2} = -0.5625t^2k^2$$

The third term

$$= ZJ^{-1}\left[\frac{k\beta}{z}ZJ[0.28125kt]\right] = ZJ^{-1}\left[\frac{0.28125k^2}{z}\beta ZJ[t]\right]$$

$$= ZJ^{-1}\left[\frac{0.28125k^2\beta}{z}\left[\frac{\beta^{n+2}}{z^3}\right]\right] = 0.28125\,k^2t^2$$

$$y_2 = 0.375k^2t^2 - 0.5625t^2k^2 + 0.28125k^2t^2$$

$$y_2 = 0.09375k^2t^2 \;\; andthe$$

Solution is

$$\boldsymbol{y = 0.5 + 0.375kt + 0.09375k^2t^2} + \cdots$$

Now the exact solution is

$$y(t) = 1 - \frac{1}{\sqrt{1 + 3e^{2kt}}}$$

Checking with k=1 and t=0.1

With $\mathbf{y(0.1) = 0.53543}$ and $\mathbf{y(0.1) = 1 - \frac{1}{\sqrt{1+3e^{2(0.1)}}} \approx 0.538}$ It's right

Conclusions

The Adomian-ZJ method for second-order nonlinear ODEs implies:

1. Apply the ZJ transform to the ODE and initial conditions.Solve algebraicallyforZJ ($\hat{\varphi}$)in terms of L{N(u)}.
2. Decompose the solution ZJ ($\hat{\varphi}$)and the nonlinear operator L{N(u)}in series.
3. Determine the components$\hat{\varphi}_n(x,t)$recursively using the transformed Adomian polynomials.
4. Apply the inverse ZJ transform to each $\hat{\varphi}_n(x,t)$ to obtain the components $\hat{\varphi}_n(x,t)$of the solution in the original domain.
5. Approximatethesolutionu(x, t) byaddingthefirstcomponentsobtained.

The combination of the Adomian Decomposition Method (ADM) with the ZJ transform (Adomian-ZJ method) as well as the Adomian Laplace method offers several significant advantages over classical methods for solving certain nonlinear ODEs:

1. Does not require linearization or perturbation:
- Classical methods for nonlinear equations often rely on linearizing the problem or applying perturbation techniques that assume the presence of a small parameter. These approximations can be restrictive and may not be valid for problems with strong nonlinearities or without evidently small parameters.

2. Provides solutions in the form of convergent series:
- The MDA searches for the solution as an infinite series whose components are determined recursively. If the series converges, it provides an approximate analytical solution that can approach the exact solution with sufficient terms.

3. Avoids discretization (unlike numerical methods):
- Many classical methods for nonlinear equations, especially when analytical solutions cannot be found, resort to numerical techniques that involve discretization of the domain. This introduces discretizationerrors and can be computationally expensive.

However, it is important to keep in mind some limitations:
- The convergence of the Adomian series is not always guaranteed and may depend on the nature of the nonlinear equation and the problem conditions.
- Calculating Adomian polynomials for complex nonlinearities can be laborious.

The Adomian-ZJ method offers a powerful and elegant alternative to classical methods for solving certain nonlinear ODEs, especially those where linearization or perturbations are unsuitable or where an approximate analytical solution is sought without resorting to discretization. Its ability to handle nonlinearities directly and the simple incorporation of initial conditions are key advantages.

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