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On Semitopological Lattice

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Abstract: In this paper we make a study of semitopological lattices a notion weaker than well known one of topological lattices. A topological lattice is always a semitopological lattices but the converse is not true as shown by an example. We drive here conditions that imply a semitopological lattices is a topological lattice.
Keywords: Topological Lattices, Semitopological Lattice, Homeomorphisms, Homogeneous Lattice.

Introduction

A semitopological lattice is and algebraic structure (L, $\wedge$, $\vee$) equipped with a topology such that meet ($\wedge$) and join ($\vee$) operations are separately continuous meaning that for each fixed element, the operation is continuous with respect to the other variable. Unlike a **topological lattice**, where both operations are jointly continuous a semi topological lattice imposes a weaker requirement making it useful in situations where full continuity is not achievable but partial continuity still holds.

It arise in the study of ordered topological structures providing a link between pure lattice theory and topology and is applied in areas like functional analysis domain theory and theoretical computer science.

A semitopological lattice is a lattice equipped with topology where the lattice operations (meet $\wedge$ and join $\vee$) are separately continuous that is continuous in each variable when the other is fixed.

G. Birkhoff (p. 248, AMS Publication, reprinted 1984) defined a topological lattice as a lattice with specified convergence topology in which

(i) $x_\alpha \to x, y_\beta \to y \Rightarrow x_\alpha \wedge y_\beta \to x \wedge y$

(ii) $x_\alpha \to x, y_\beta \to y \Rightarrow x_\alpha \vee y_\beta \to x \vee y$

The Concept of Semitopological Lattices

Definition 1: (a) A topological lattice L is also a lattice which is also called a semitopological lattice if the mapping is continuous in both variables together if the mapping

$$g_1 : (x, y) \to x \vee y$$

of $L \times L$ onto L is continuous in each variable separately.

(b) A topological lattice L is also a lattice which is called a topological lattice if the mapping is continuous in both variables together if the inversion mapping

$$g_2 : (x, y) \to -x$$

of $L \times L$ onto L is continuous.

Theorem 1: Every topological lattice is a semitopological lattice. But the converse is not true.

Proof: The first statement is clearly true. To show that the converse is not true. Let $L = R$. the set of real numbers endowed with a topology which has $\{[a, b): -\infty < a < x < b < \infty\}$, the system of left closed and right open intervals as its base. Since for each neighbourhood $[a, b)$ of the number 0, $\left[a, \frac{b}{2}\right]$ is also a neighbourhood of 0, it follows that the mapping g, is continuous in both variables together at 0. It is easy to see that g_1 is continuous everywhere. Hence L is a semitopological lattice. However, the mapping g_2: $x \to -x$ is not continuous at 0 because if [0, b) is a neighbourhood 0, then there is no neighbourhood V of 0 such that $-V \subseteq$ [0, b) Therefore L is not a topological lattice. This completes the proof.

It is easy to see that the mappings g_1 and g_2 are continuous in all their variables together if, and only if, the mapping $g_3 :(x, y) \to x \wedge (-y)$ of $L \times L$ onto L is continuous.

Theorem 2: Let a be a fixed element of a semitopological lattice *L*. Then the mapping

$$\gamma_a : x \to x \wedge a$$
$$l_a : x \to a \wedge x$$

of *L* onto *L* are homeomorphisms of *G*.

Proof: It is clear that γ_a is a $|:|$ and onto mapping. Let *W* be a neighbourhood of x $\wedge$ a. Since *L* is semitopological lattice, there exists a neighbourhood *U* of *x* such that $U_\alpha = U \wedge a \subseteq W.$ This show that γ_a is continuous.

Moreover, it is easy to see that the inverse of γ_a is the mapping x $\to$ x $\wedge$ (– a) which is continuous by the same argument as above. Hence γ_a is a homeomorphism. The fact l_a is a homeomorphism follows similarly.

γ_a and l_a are respectively called the right and the left translations of *L*.

Corollary 1: Let *F* be closed, *P* an open, and *A* any subset of a semitopological lattice *L* and let $a \in L$ Then

(i) $F_a \to F \wedge a$ and $aF = a \wedge F$ are closed.

(ii) $P_a \to P \wedge a$ and $ap = a \wedge P,$

AP and *PA* are open.

Proof: Since the mapping in theorem 2 are homeomorphisms, (i) is obvious. By the same argument, *Pa* and *aP* are open in (ii).

Since $AP = \underset{a \in A}{V}(a \wedge p : p \in P)$

$$PA = \underset{a \in A}{V}(p \wedge a : p \in P)$$

are supremum of open sets is open.

Corollary 2: Let *L* be a semitopological lattice. For $x_1, x_2 \in L$, there exists a homeomorphisms *f* of *L* such that $f(x_1) = x_2$

Proof: Let $(-x_1) \wedge x_2 = a \in L$ and consider the mapping $f{:}x \to x \wedge a$. Then *f* is a homeomorphism by theorem 1 and $f(x_1) = x_2$. A lattice for which Corollary 2 is true is called homogeneous lattice.

Conclusion:

A semitopological lattice provides a weaker structure then a polopogical lattice, offering flexibility when full joint continuity of operations not required. It surves as an important framework in the study of ordered topological structures, where algebraic and topological properties interact and can be stepping stone toward understanding topological lattices, domain theory and related algebraic topological systems.

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