International Journal of Latest Research in Engineering and Technology (IJLRET)
ISSN: 2454-5031
www.ijlret.com || Volume 11 - Issue 10 || October 2025 || PP. 29-33

Strictly U-Pseudo Regular U-spaces

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Abstract: In this paper Strictly U -pseudo regular U- spaces have been defined and a few important properties of these spaces have been justified. This is the fourth paper of U- Pseudo regular U- spaces. A number of important theorems regarding this space have been established. Strictly U -pseudo normal is not defined for U-spaces.

Keywords: U- space, U-Regular, U-Compact, U-Continuous, Strictly U-Pseudo Regular U-Spaces
Mathematics Subject Classification: 54D10, 54D15, 54A05, 54A40, 54G12, 54C08

I. Introduction

A topological space X is said to be **regular** if for each non-empty closed set F of X and any point x $\in$ X, such that X $\notin$ F (i.e. x is the external point of F), there exist two disjoint open sets V and W such that x $\in$ V and F $\subseteq$ W. In Paper [3] a pseudo regular topological spaces has been defined by replacing a closed subset F by a compact subset K in the definition of a regular space. The concept of U- Pseudo regularity has been extended to U- spaces in previous paper [2]. These spaces were introduced and studied in [4] and [5]. Here the concept of Strictly U-Pseudo regularity of [1] has been extended to U-spaces. In this paper we have given examples of Strictly U-pseudo regular U- spaces. A number of important theorems regarding these spaces have been established.

Here we have introduced strictly pseudo regular U- spaces and studied their important properties. Many results have been proved about these U-spaces. Strictly U- pseudo normal U –spaces is not established. We have also established characterizations of such U-spaces.

II. Preliminaries

We begin with some basic definitions and examples related to U– spaces.

U-Spaces

Definition 2.1[5]: A U-**structure** on a non empty set X is a collection $\mathcal{U}$ of subsets of X having the following properties: (i) Φ and X are in $\mathcal{U}$, (ii) Any union of members of $\mathcal{U}$ is in $\mathcal{U}$.

The ordered pair (X, $\mathcal{U}$) is called a **U-space**. A U-space which is not a topological space is called a **proper U-space**. The members of $\mathcal{U}$ are called **U-open set** and the complement of a U-open set is called a **U-closed set**.

A U- structure and a U-space have been called a supra-topology and a supra-topological space respectively by some authors (see [6], [7], [8], [9])

In general we have
Topological space $\Rightarrow$ U-space, Topological space $\not\Leftarrow$ U-space

Example 2.1: Let X ={a, b, c, d}, $\mathcal{U}$ ={X, Φ, {a, b},{a, c}, {a, b, c}}. **Here (X, $\mathcal{U}$) is a U-space but nota topological space.**

Definition 2.2 [4]: Let (X, $\mathcal{U}$) be a U-space. A **U-open cover** of a subset K of X is a collection $\{G_\alpha\}$ of U-open Sets such that K $\subseteq$ $\cup_\alpha G_\alpha$.

Definition 2.3 [4]: A U-space X is said to be **U-compact** if for every U-open cover of X has a finite sub-cover.

A subset K of a U-space X is said to be **U-compact** if every U-open cover of K has finite sub-cover.

Example 2.2: Let X = **N**, $\mathcal{U}$ = {**2N, 4N, 8N, 16N,**............................**2^n N,......... ,N**,Φ}. Then X is U-compact.

Let $\Phi \neq A \subseteq X$ and **C** be a U open cover of A. Let n_0 be smallest + ve integer such that 2^{n0} **N** $\in$ **C.**
Then $A \subseteq 2^{n0}$**N.** So $\{2^{n0}$ **N**$\}$ is a finite sub-cover of **C.** Therefore any subset K of X, $K \neq X$ is U- compact.

Definition 2.4 [4]: A U-space X is called **Hausdorff** if, for each x, y $\in$ X, x $\neq$y, there exists disjoint U- open sets G and H in X such that $x \in G$, $y \in H$.

Example 2.3: Let X = {a, b, c, d}, $\mathcal{U}$ = {{a},{d},{b, c},{b, d},{a, d},{a, c},{a, b, c},{b, c, d},{a, c, d},{a, b, d}, X, ϕ}.Then (**X**, $\mathcal{U}$) is a Hausdorff U-space.

Definition 2.5 [4]: A U- space X is called **U- regular space** if for any U- closed set F of X and any point $x \in X$, such that $x \notin F$ (i.e. x is the external point of F) there exist two disjoint U-open sets G and H such that $x \in G$ and $F \subseteq H$.

For a U-space, 'Hausdorff' and regular are independent concepts.

Example of a U-space which is regular but not U-Hausdorff space

Example 2.4: Let X = {a, b, c, d}, $\mathcal{U}$ = {X, ϕ, {a}, {d}, {a, d}, {a, b, c}, {b, c, d}}. Then (X, $\mathcal{U}$) is a U-space. Here the U-closed sets are X, ϕ, {a}, {d}, {b, c}, {a, b, c}, {b, c, d}. We easily sec that X is a U- regular space but it is not U- Hausdorff space, since b and c cannot be separated by disjoint U- open sets. Also X is not a topological space.

Definition 2.6 [4]: A U- space X is said to be U- **completely regular** if for any U- closed subset F of X and $x \in X$ which does not belongs to F, there exists a U- continuous function f: $X \rightarrow [0,1]$ such that f(x) = 0 and f(F) = 1. Here [0, 1] is considered as a sub space of the usual U-space R.

Example 2.5: The set of real numbers R is U-completely regular

Definition 2.7: Let **(X,** $\mathcal{U}$**)** and **(Y,** $\mathcal{U}_1$**)** be two U- spaces. A function f: $X \rightarrow Y$ is said to be **U- continuous** if for each U-open set Gin Y, $f^{-1}(G)$ is U-open set in X.

Example 2.6: Let X= {a, b, c, d }, $\mathcal{U}$= {X, ϕ, {a}, {a, b}, {a, c, d}, {b, c, d}}
Y = {p, q, r}, $\mathcal{U}_1$ = {Y, ϕ, {p}, {p, q}, {p, r}, {q, r}}. Let f: $X \rightarrow Y$ be defined by f(a) = p, f(b) = q, f(c) = r, f(d) = r. Then f is U- continuous.
Here **(X,** $\mathcal{U}$ **)** and **(Y,** $\mathcal{U}_1$ **)** are two U-spaces but not a topological spaces.

Definition 2.8 [4]: A U-space X is said to be **U- Normal Space** if for each paird is joint U-closed sets F_1 and F_2, there exist U- open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \phi$.

Comments: **Generally U-regular space is not U-normal space and U-normal space is not U-regular space. Generally U- pseudo regular space is not U- pseudo normal space.**

Example 2.7: Let X={a, b, c, d}, U ={X, ϕ, {a}, {a, b}, {c, d}, {a, c, d}, {b, d}, {a, b, d}, {b, c, d}}. Closed sets are X, ϕ, {a}, {a, b}, {c, d}, {c}, {b}, {a, c}, {b, c, d}.
Here {a, b}$\subseteq${a, b}and{c, d}$\subseteq${c d}. $\therefore$ X is U-normal space. Here {c, d} is closed set, $b \notin$ {c, d}.
But there does not exist disjoint U-open sets containing band {c, d} respectively.

Pseudo Regular U- Spaces

Definition 2.9[2]: A X U-space will be called **pseudo regular** if for every U-compact subset K of X and for every $x \in X$, $x \notin K$, there exist U-open sets G_1 and G_2 in X with $G_1 \cap G_2 = \phi$ such that $x \in G_1$ and $K \subseteq G_2$.

Example 2.8: Let X= Z, and The U-structure generated by $\{\{Z,\phi\} \cup \{(-\infty,a)]|a \in Z\} \cup \{[(b,\infty)|b \in Z\}\}$, $\mathcal{U} = <\{\{Z,\phi\} \cup \{(-\infty,a)]\ |a \in Z\ \} \cup \{[(b,\infty)|b \in Z\ \}\}$.**Then X is pseudo regular U-space.**

Proof: The subsets of X are: $(-\infty, a)]$, $a \in Z$.....(1), $[(b,\infty)$, $b \in Z$.....(2),$[(c, d)]$, $c, d \in Z$......(3).
The sets in (3) are finite, and so, compact.
If $A = (-\infty, a)]$, *for some* $a \in Z$, then any U-open cover C of A must contain $(-\infty, a')]$ where $a \le a'$. Then $\{(-\infty, a')]\}$ is a finite sub cover of C. Thus, the sets in (1) are compact.
Similarly, we can show that the sets in (2) are compact.
Thus, every non empty subset of X is compact.

Let K be a compact subset of X and $x \in X$ with $x \notin K$.
Case-I: If K is a set of the form (1) i.e., $K = (-\infty, a)]$, *for some* $a \in Z$, then $x \in Z$, $x > a$, ($x \geq a$ *if* $K = (-\infty, a)$))
Let choose $G_1 = [(a+1, \infty)$ if $K = (-\infty, a]$, $G_1 = [a, \infty)$ if $K = (-\infty, a)$, and $G_2 = (-\infty, a)]$ if $K = (-\infty, a]$, $G_2 = (-\infty, a)$ if $K = (-\infty, a)$. Thus, in each case $x \in G_1$, $K \subseteq G_2$ and $G_1 \cap G_2 = \phi$.

Case-II: If K is a set of the form (2), i.e. $K = [(b, \infty)$, for some $b \in Z$, then $x \le b$, or $x < b$, according as $K = (b, \infty)$, or $[b, \infty)$.

Let choose $G_1 = (-\infty, b]$, $G_2 = (b, \infty)$, if $K = (b, \infty)$; $G_1 = (-\infty, b)$, $G_2 = [b, \infty)$ if $K = [b, \infty)$
Then for each case, $x \in G_1$, $K \subseteq G_2$ and $G_1 \cap G_2 = \phi$.

Case-III: Now let $K = [(c, d)]$, for some $c, d \in Z$. Then, (i) $x \le c$ and / or $x \ge d$, if $K = (c, d)$,
(ii) $x < c$, and/ or $x \geq d$, if $K = [c, d)$, (iii) $x \le c$, and / or $x > d$, if $K = (c, d]$, (iv) $x < c$, and / or $x > d$, if $K = (c, d]$
For (i), we choose $G_1 = (-\infty, c] \cup (d, \infty)$, $G_2 = (c, d)$, for (ii), we choose $G_1 = (-\infty, c) \cup [d, \infty)$, $G_2 = [c, d)$, For (iii), we choose $G_1 = (-\infty, c] \cup (d, \infty)$, $G_2 = (c, d]$, for (iv), we choose $G_1 = (-\infty, c) \cup (d, \infty)$, $G_2 = [c,d]$.
Then, for each case $x \in G_1$, $K \subseteq G_2$ and $G_1 \cap G_2 = \phi$. **Thus, X is pseudo regular U-space.**

Example 2.9: **A U–space X may be regular but not pseudo regular.**
Proof: Let $X = \{a, b, c, d\}$ and $U = \{X, \phi, \{a, b\}, \{c, d\}, \{a, c, d\}, \{a\}\}$.
Here closed sets are $\{c, d\}$, $\{a, b\}$, $\{b, c, d\}$, $\{b\}$, X, ϕ. Then (X, U) is U- regular. For, $\{a, c\}$ is compact, $b \notin \{a, c\}$, but $\{a, c\}$ and b cannot be separated by disjoint U-open sets. Here (X, U) is not pseudo regular.

III. Strictly U- Pseudo Regular U- Spaces

Definition 3.1: A U – space X will be called **Strictly U-pseudo-regular** if for each U- compact set K and for every $x \in X$ with $x \notin K$, there exists a continuous function $f : X \to [0, 1]$ such that f (x) = 0 and f (K) = 1.

Example 3.1: Let K be a U- compact subset of R and let $x \in R$ such that $x \notin K$. Since R is U– Hausdorff, K is closed and since R is U-completely regular, there exists a continuous function f: $X \to [0, 1]$ such that f(x)=0 and f (K) = 1. Thus R is strictly U- pseudo regular.

Theorem 3.1 [1]: Every strictly U-pseudo regular compact space is U-completely regular.

Proof: Let X be U-compact and strictly U-pseudo-regular. Let K be a closed subset of X and let $x \in X$ with $x \notin K$. Since X is U-compact, K is U-compact. Again, since X is strictly U–pseudo regular, there exists a U-continuous function f: $X \to [0, 1]$ such that f(x) = 0 and f(K) = 1. Therefore X is U- completely regular.

Theorem 3.2[1]: Every U-completely regular Hausdorrf space is strictly U-pseudo regular.

Proof: Let X be a U-completely regular Hausdorrf space. Let K be a U-compact subset of X and $x \in X$ with $x \notin K$. Since X is U- Hausdorrf, K is U-closed. Now, since X is U–completely regular, there exists a U- continuous function f: $X \to [0, 1]$ such that f(x) = 0 and f(K) = 1. Therefore X is strictly U- pseudo regular.

Theorem 3.3 [1]: A U-space X is strictly U- pseudo regular if for each $x \in X$ and any U- compact set K not containing x, there exists an U-open set H of X such that $x \in H \subseteq \overline{H} \subseteq K^C$.

Proof: Let X be a strictly U pseudo regular space and Let K be U-compact in X. Let x∉K i.e., $x \in K^C$. Since X is strictly U-pseudo regular, there exists a U-continuous function f:X → [0,1] such that f(x)=0 and f(K)=1. Let a, b∈ [0,1] and a< b. Then [0, a) and (b, 1] are two disjoint open sets of [0, 1]. Since f is U-continuous $f^{-1}([0, a))$ and $f^{-1}((b, 1])$ are two disjoint open sets of X and obviously $x \in f^{-1}([0, a))$ and $K \subseteq f^{-1}((b, 1])$.
Let $U = f^{-1}([0, a))$ and $V = f^{-1}((b, 1])$. Then $x \in U$, $K \subseteq V$ and $U \cap V = \phi$. Then $U \subseteq V^C \subseteq K^C$.
So $U \subseteq V^C = V^C \subseteq K^C$. Writing U = H, we have $x \in H \subseteq H \subseteq K^C$.

Theorem 3.4[1]: Any subspace of a strictly U-pseudo regular space is strictly U-pseudo–regular.

Proof: Let X be a strictly U-pseudo regular space and $Y \subseteq X$.. Let $y \in Y$ and K be a U-compact subset of Y Such that $y \notin K$. Since $y \in Y$, so $y \in X$ and since K is U-compact in Y, so K is U-compact in X. Since X is strictly U- pseudo regular, there exists a continuous function f : X → [0, 1] such that f(y) = 0 and f(K) = 1. Therefore the restriction function *f* of f is a continuous function *f*: Y → [0, 1] such that f(y) = 0 and f(K) = 1. Hence Y is strictly U- pseudo regular.

Corollary: Let X be a U- space and A, B are two strictly U- pseudo regular sub space of X. Then $A \cap B$ is strictly U- pseudo regular.

Proof: Since $A \cap B$ being a subspace of both A and B, $A \cap B$ is strictly U- pseudo regular by the above theorem.

Theorem 3.5[1]: Every strictly U-pseudo regular space is U–Hausdorff.

Proof: Let X be a strictly U-pseudo regular space. Let $x, y \in X$. with $x \neq y$. Then{x} is a U–compact set and $y \notin \{x\}$. Since X is strictly U-pseudo regular, there exists a continuous function f:X→[0,1] such that f(y)=0 and f({x})=1. Let a, $b \in [0,1]$. And a< b. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is U-continuous, $f^{-1}([0,a))$ and $f^{-1}((b, 1])$ are two disjoint open sets of X and obviously $y \in f^{-1}([0,a))$ and $\{x\} \subseteq f^{-1}((b,1])$ i.e.,$x \in f^{-1}((b,1])$. Therefore X is U- Hausdorff.

Theorem 3.6 [1]: Every strictly U-pseudo regular space is U-pseudo regular.

Proof: Let X be a strictly U-pseudo regular space. Let K be a compact subset of X and $x \in X$. with $x \notin K$..
Since X is strictly U-pseudo regular, there exists a continuous function f: X→[0,1] such that f(x)=0 and f(K) =1. Let a, $b \in [0,1]$. And a< b. Then [0, a) and (b, 1] are two disjoint open sets of [0, 1]. Since f is U-continuous, $f^{-1}([0, a))$ and $f^{-1}((b, 1])$ are two disjoint open sets of X and obviously $x \in f^{-1}([0,a))$ and $K \subseteq f^{-1}((b,1])$. Therefore X is U-pseudo regular.

Result and Discussion:

It has been proved that for U- spaces, Every strictly U- pseudo regular compact space is U- completely regular; Every U- completely regular Hausdorff space is strictly U- pseudo regular; A U- space X is strictly U-pseudo regular if for each x∈X and any U- compact set K not containing x, there exists anU-open set H of X such that$x \in H \subseteq H \subseteq K^C$.;Any subspace of a strictly U- pseudo regular space is strictly U- pseudo – regular; Every strictly U- pseudo regular space is U – Hausdorff; Every strictly U- pseudo regular space is U- pseudo regular.

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