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Solution of Nonlinear Mixed Partial Differential Equations using the Adomian Decomposition Method and the ZJ Transform

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Abstract: In this new article, the Adomian Decomposition Method is presented, along with the application of the ZJ transform together for the solution of second-order nonlinear Mixed PDEs and a generalization of this type of equation through illustrative examples, the effectiveness and potential of this hybrid approach to obtain approximate analytical solutions of nonlinear problems will be analyzed.

Keywords: ZJ Transform, Adomian Decomposition Method, Adomian Polynomials, Nonlinear Differential Equations, Series, Mixed Nonlinear Partial Differential Equations.

Introduction:

Ordinary differential equations (ODEs) and nonlinear partial differential equations (PDEs) model a vast range of complex phenomena in various scientific and engineering disciplines. However, the inherently nonlinear nature of these equations often makes obtaining exact analytical solutions difficult. Consequently, the development and application of robust and efficient approximate numerical and analytical methods have become an active and crucial area of research.

Among the powerful analytical techniques for addressing ODEs as well as nonlinear PDEs , the **Adomian Decomposition Method (ADM)** has emerged as a versatile and effective tool. Introduced by George Adomian in the 1980s, ADM offers a methodology for obtaining solutions in the form of convergent series without requiring linearization, discretization, or the introduction of small perturbation parameters—features that often limit the applicability of other traditional methods. The cornerstone of ADM lies in the decomposition of the nonlinear operator present in the equation into a series of specific polynomials, known as **Adomian polynomials**, which depend on the components of the series solution.

The use of **integral transforms**, such as the **Laplace transform** , has proven invaluable in simplifying and solving differential equations, especially those with linear terms and well-defined initial conditions. Integral transforms convert a differential equation into an algebraic equation in the domain of the transform, which is often easier to solve. Subsequently, applying the inverse transform allows us to obtain the solution in the original domain.

The strategic combination of the Adomian Decomposition Method with integral transform techniques has led to promising hybrid approaches for treating nonlinear ODEs and PDEs. By applying an integral transform to the original differential equation, the linear terms can be simplified, while the Adomian Decomposition Method handles the nonlinear part by generating the corresponding Adomian polynomials.

General Description of the Combined Method (Adomian-ZJ) for Second Order Nonlinear Mixed PDEs

Let us consider a second-order nonlinear PDE of the general form:

$$L\big(u(x,t)\big) + R\big(u(x,t)\big) + N\big(u(x,t)\big) = h(x,t)$$

where:

- L(u(x,t)) = $\frac{\partial^2}{\partial x \partial y}$the linear operator of order 2
- R (u(x, t)) represents the linear operator with first-order derivatives.
- N (u(x, t)) represents the nonlinear operator.
- h (x, t) is the non-homogeneous function (source term).

Thus, the EDP is:

$$\frac{\partial^2 u}{\partial x \partial y} + R\big(u(x,t)\big) + N\big(u(x,t)\big) = h(x,t)$$

With U =$\frac{\partial u}{\partial y}$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)+\ R\big(u(x,t)\big)+N\big(u(x,t)\big)=h(x,t)$$

$$\frac{\partial U}{\partial x}+\ R\big(u(x,t)\big)+N\big(u(x,t)\big)=h(x,t)$$

$$u(x,0)=f(x)\,\frac{\partial u(0,y)}{\partial y}=g(y)$$

Now taking the ZJ transform on both sides of the equation with respect to x

Step 1: Application of the ZJ Transform

We apply the ZJ transform, denoted by ZJ { · } , to both sides of the PDE:

$$\text{ZJ}\left\{\frac{\partial U}{\partial x}\right\}+\text{ZJ}\{R(u(x,t))\}+\text{ZJ}\{N(u(x,t))\}=\text{ZJ}\{h(x,t)\}$$

Using the properties of the ZJ transform for first-order derivatives:

$$\text{ZJ}\left\{\frac{\partial U}{\partial x}\right\}=\frac{z}{\beta}\widehat{U(x,y)}-U(0,y)\frac{\beta^n}{z}$$

Step 2: Solution in the ZJ Domain by accommodating terms within the PDE

$$\widehat{U(x,y)}=\frac{\beta^{n+1}}{z^2}\,U(0,y)-\frac{\beta}{z}ZJ_x[Ru]-\frac{\beta}{z}ZJ_x[Nu]+\frac{\beta}{z}ZJ_x[h]$$

Replacing the boundary condition

$$\widehat{U(x,y)}=\frac{\beta^{n+1}}{z^2}\,g(y)-\frac{\beta}{z}ZJ_x[Ru]-\frac{\beta}{z}ZJ_x[Nu]+\frac{\beta}{z}ZJ_x[h]$$

Now taking the inverse ZJ transform with respect to x

$$\boldsymbol{U(x,y)}=\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta^{n+1}}{z^2}\,g(y)\right]-\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[h-Ru-Nu]\right]$$

$$\boldsymbol{U(x,y)}=g(y)-\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[h-Ru-Nu]\right]$$

Now we have a nonlinear, nonhomogeneous, first-order PDE, taking the ZJ transform with respect to y

$$\boldsymbol{ZJ_y\left[\frac{\partial u}{\partial y}\right]=ZJ_y[g(y)]-ZJ_y\left[ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x[h-Ru-Nu]\right]\right]}$$

Now with

$$\text{ZJ}\left\{\frac{\partial u}{\partial y}\right\}=\frac{z}{\beta}\widehat{u(x,y)}-u(x,0)\frac{\beta^n}{z}$$

$$\frac{z}{\beta}\widehat{u(x,y)}-u(x,0)\frac{\beta^n}{z}=\frac{z}{\beta}\widehat{u(x,y)}-\frac{\beta^n}{z}f(x)$$

So now we have

$$\widehat{u(x,y)}=\frac{\beta^{n+1}}{z^2}\,f(x)+\frac{\beta}{z}ZJ_y[g(y)]+\frac{\beta}{z}ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[h-Ru-Nu]\right]\right]$$

Now taking the inverse ZJ transform with respect to y and

$$\boldsymbol{u(x,y)}=\boldsymbol{ZJ_y^{-1}}\left[\frac{\beta^{n+1}}{z^2}\,f(x)\right]+\boldsymbol{ZJ_y^{-1}}\left[\frac{\beta}{z}ZJ_y[g(y)]\right]+\boldsymbol{ZJ_y^{-1}}\left[\frac{\beta}{z}ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[h-Ru-Nu]\right]\right]\right]$$

The indicated transformation expression is obtained.

$$\boldsymbol{u(x,y)=f(x)+ZJ_y^{-1}\left[\frac{\beta}{z}ZJ_y[g(y)]\right]+ZJ_y^{-1}\left[\frac{\beta}{z}ZJ_y\left[ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x[h-Ru-Nu]\right]\right]\right]}$$

Step 3: Application of the Adomian Decomposition Method

Now, we apply the MDA to solve the equation in the ZJ domain. We asume that the solutionu (t) can be expressed as an infinite series. super position principle, the solution can be represented as an infinite series$\sum_{n=0}^{\infty} \hat{\varphi}_n(x,t)$

If we observe the nonlinear operator N u (x, t) from our equation, we decompose it as a series of Adomian polynomials as N u (x, t) = $\sum_{n=0}^{\infty} A_n(x,t)$where An are Adomian polynomials of $u_n$and it can be determined by the relation

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\sum_{i=0}^{\infty}\lambda^n u_i\right]$$

Now calculating the Adomian Polynomials

$$A_0 = u_0^2$$
$$A_1 = 2u_0u_1$$
$$A_2 = 2u_0u_2 + u_0^2$$

With

$$\left(\frac{\partial u_0}{\partial y}\right)^2 = A_0$$

Application Examples

Example 1

Let the Mixed Nonlinear Second Order Partial Differential Equation be

$$\frac{\partial^2 u}{\partial x \partial y} - \left(\frac{\partial u}{\partial y}\right)^2 = 0$$

With $u(x,0) = 0 \quad \frac{\partial u(0,y)}{\partial y} = 1$

By $U = \frac{\partial u}{\partial y}$ applying the ZJ Transform, on both sides

$$ZJ\left[\frac{\partial U}{\partial x}\right] - ZJ\,[U^2] = 0$$

Now, regarding the properties of the first derivative of the Z-transform, **step 1**

$$\text{ZJ}\left\{\frac{\partial U}{\partial x}\right\} = \frac{z}{\beta}\widehat{U(x,y)} - U(0,y)\frac{\beta^n}{z}$$

Step 2

Replacing the boundary condition

$$\widehat{U(x,y)} = \frac{\beta^{n+1}}{z^2} + \frac{\beta}{z}ZJ_x[U^2]$$

Now taking the inverse ZJ transform with respect to x

$$U(x,y) = ZJ_x^{-1}\left[\frac{\beta^{n+1}}{z^2}\right] + ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x[U^2]\right]$$

$$U(x,y) = 1 + ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x[U^2]\right]$$

Now we have a nonlinear, nonhomogeneous, first-order PDE, taking the ZJ transform with respect to y

$$ZJ_y\left[\frac{\partial u}{\partial y}\right] = ZJ_y[1] + ZJ_y\left[ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right]$$

So now we have

$$\widehat{u(x,y)} = \frac{\beta^{n+2}}{z^3} + \frac{\beta}{z}ZJ_y\left[ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right]$$

Now taking the inverse ZJ transform with respect to y and

$$u(x,y) = ZJ_y^{-1}\left[\frac{\beta^{n+2}}{z^3}\right] + ZJ_y^{-1}\left[\frac{\beta}{z}ZJ_y\left[ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right]\right]$$

$$\boldsymbol{u(x,y) = y + ZJ_y^{-1}}\left[\frac{\beta}{z}ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x\left[\left(\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial y}}\right)^2\right]\right]\right]\right]$$

Step 3: Application of the Adomian Decomposition Method The System remains as

$$u_0 = y$$

$$u_{n+1} = \boldsymbol{ZJ_y^{-1}}\left[\frac{\beta}{z}ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x\left[\left(\frac{\boldsymbol{\partial u_0}}{\boldsymbol{\partial y}}\right)^2\right]\right]\right]\right]$$

Now calculating the Adomian Polynomials

$$A_0 = u_0^2$$
$$A_1 = 2u_0u_1$$
$$A_2 = 2u_0u_2 + u_0^2$$

With

$$\left(\frac{\partial u_0}{\partial y}\right)^2 = A_0$$

Thus with n = 0

$$u_1 = xy$$

Thus with n = 1

$$A_1 = 2u_0u_1 = 2\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y} = 2x$$
$$u_2 = x^2y$$

Therefore, when forming the series we have

$$u_n = u_0 + u_1 + u_2 + \cdots$$
$$u_n = y + xy + x^2y + \cdots$$

The series converges thus and finally we have the solution analytics.

$$u(x,y)_n = y\sum_{n=0}^{\infty} x^n = \frac{y}{1-x}$$

Example 2
Let the Mixed Nonlinear Second Order Partial Differential Equation be

$$\frac{\boldsymbol{\partial^2 u}}{\boldsymbol{\partial x \partial y}} + \left(\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial y}}\right)^2 = \mathbf{1}$$

With$\boldsymbol{u(x,0) = 0} \quad \frac{\boldsymbol{\partial u(0,y)}}{\boldsymbol{\partial y}} = \mathbf{0}$

by $U = \frac{\partial u}{\partial y}$ applying the ZJ Transform, on both sides

$$\boldsymbol{ZJ}\left[\frac{\boldsymbol{\partial U}}{\boldsymbol{\partial x}}\right] + \boldsymbol{ZJ}\,[\boldsymbol{U^2}] = \boldsymbol{ZJ}[\mathbf{1}]$$

Now, regarding the properties of the first derivative of the Z-transform, **step 1**

$$\text{ZJ}\left\{\frac{\partial U}{\partial x}\right\} = \frac{z}{\beta}\widehat{U(x,y)} - U(0,y)\frac{\beta^n}{z}$$

Step 2
Replacing the boundary condition

$$\widehat{U(x,y)} = \frac{\beta^{n+2}}{z^3} - \frac{\beta}{z}ZJ_x[U^2]$$

Now taking the inverse ZJ transform with respect to x

$$\boldsymbol{U(x,y) = ZJ_x^{-1}}\left[\frac{\beta^{n+2}}{z^3}\right] - \boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[U^2]\right]$$

$$\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial y}} = x - \boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[U^2]\right]$$

Now we have a nonlinear, nonhomogeneous, first-order PDE, taking the ZJ transform with respect to y

$$\boldsymbol{ZJ_y\left[\frac{\partial u}{\partial y}\right] = ZJ_y[x] - ZJ_y\left[ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right]}$$

So now we have

$$\widehat{u(x,y)} = x\frac{\beta^{n+2}}{z^3} - \frac{\beta}{z}ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x\left[\left(\boldsymbol{\frac{\partial u}{\partial y}}\right)^2\right]\right]\right]$$

Now taking the inverse ZJ transform with respect to y and

$$\boldsymbol{u(x,y) = xZJ_y^{-1}}\left[\frac{\beta^{n+2}}{z^3}\right] - \boldsymbol{ZJ_y^{-1}}\left[\frac{\beta}{z}ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x\left[\left(\boldsymbol{\frac{\partial u}{\partial y}}\right)^2\right]\right]\right]\right]$$

$$\boldsymbol{u(x,y) = yx - ZJ_y^{-1}}\left[\frac{\beta}{z}ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x\left[\left(\boldsymbol{\frac{\partial u}{\partial y}}\right)^2\right]\right]\right]\right]$$

Step 3: Application of the Adomian Decomposition Method The system remains as

$$u_0 = yx$$

$$u_{n+1} = -\boldsymbol{ZJ_y^{-1}}\left[\frac{\beta}{z}ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x\left[\left(\boldsymbol{\frac{\partial u_0}{\partial y}}\right)^2\right]\right]\right]\right]$$

Now calculating the Adomian Polynomials

$$A_0 = u_0^2$$
$$A_1 = 2u_0u_1$$
$$A_2 = 2u_0u_2 + u_0^2$$

With

$$\left(\frac{\partial u_0}{\partial y}\right)^2 = A_0$$

Thus with n = 0

$$u_1 = -\frac{x^3y}{3}$$

Thus with n = 1

$$A_1 = 2u_0u_1 = 2\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y} = -\frac{2x^4}{3}$$

$$u_2 = \frac{2x^5y}{15}$$

Thus with n = 2

$$u_3 = -\frac{17x^7y}{315}$$

So the series that begins to have

$$u(x,y) = yx - \frac{x^3y}{3} + \frac{2x^5y}{15} - \frac{17x^7y}{315} + \cdots$$
$$u(x,y) = y\,tanh(x)$$

Example 3
Let the Mixed Nonlinear Second Order Partial Differential Equation be

$$\boldsymbol{\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial y}\right)^2 = 1}$$

$$\textbf{With}\boldsymbol{u(x,0) = 0 \quad u(0,y) = 0 \quad \frac{\partial u(0,y)}{\partial y} = 0}$$

by $U = \frac{\partial u}{\partial y}$ applying the ZJ Transform, on both sides

$$ZJ\left[\frac{\partial U}{\partial x}\right]+ZJ\left[\frac{\partial u}{\partial x}\right]-ZJ\left[U^2\right]=ZJ[1]$$

Now, regarding the properties of the first derivative of the Z-transform, **step 1**

$$\text{ZJ}\left\{\frac{\partial U}{\partial x}\right\}=\frac{z}{\beta}\widehat{U(x,y)}-U(0,y)\frac{\beta^n}{z}$$

Step 2

Replacing the boundary condition

$$\widehat{U(x,y)}=-\widehat{U(x,y)}'+\frac{\beta^{n+2}}{z^3}+\frac{\beta}{z}ZJ_x[U^2]$$

Now taking the inverse ZJ transform with respect to x

$$U(x,y)=-ZJ_x^{-1}\left[\widehat{U(x,y)}'\right]+ZJ_x^{-1}\left[\frac{\beta^{n+2}}{z^3}\right]+ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x[U^2]\right]$$

$$\frac{\partial u}{\partial y}=-u(x,y)+x+ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x[U^2]\right]$$

Now we have a nonlinear, nonhomogeneous, first-order PDE, taking the ZJ transform with respect to y, and taking the other boundary condition

$$ZJ_y\left[\frac{\partial u}{\partial y}\right]=ZJ_y[x]+ZJ_y\left[ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right]-ZJ_y[u]$$

So now we have

$$\widehat{u(x,y)}=x\frac{\beta^{n+2}}{z^3}+\frac{\beta}{z}ZJ_y\left[ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right]-\frac{\beta}{z}ZJ_y[u]$$

Now taking the inverse ZJ transform with respect to y and

$$u(x,y)=x\,ZJ_y^{-1}\left[\frac{\beta^{n+2}}{z^3}\right]-ZJ_y^{-1}\left[\frac{\beta}{z}ZJ_y\left[ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right]\right]-ZJ_y^{-1}\left[\frac{\beta}{z}ZJ_y[u]\right]$$

$$u(x,y)=yx+ZJ_y^{-1}\left[\frac{\beta}{z}ZJ_y\left[ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right]\right]-ZJ_y^{-1}\left[\frac{\beta}{z}ZJ_y[u]\right]$$

Step 3: Application of the Adomian Decomposition Method The System remains as

$$u_0=yx$$

$$u_{n+1}=ZJ_y^{-1}\left[\frac{\beta}{z}ZJ_y\left[ZJ_x^{-1}\left[\frac{\beta}{z}ZJ_x\left[\left(\frac{\partial u_0}{\partial y}\right)^2\right]\right]\right]\right]-ZJ_y^{-1}\left[\frac{\beta}{z}ZJ_y[u]\right]$$

Now calculating the Adomian Polynomials

$$A_0=u_0^2$$
$$A_1=2u_0u_1$$
$$A_2=2u_0u_2+u_0^2$$

With

$$\left(\frac{\partial u_0}{\partial y}\right)^2=A_0=x^2$$

Thus with n = 0

$$u_1=\frac{x^3y}{3}-\frac{xy^2}{2}$$

Thus with n = 1

$$u_2=\frac{2x^5y}{15}-\frac{x^3y^2}{2}+\frac{xy^3}{6}$$

Thus, the series that begins to have is, as a solution

$$u(x,y) = yx + \frac{x^3y}{3} - \frac{xy^2}{2}\frac{2x^5y}{15} - \frac{x^3y^2}{2} + \frac{xy^3}{6} + \cdots$$

Now, as you can see, we can make a generalization about this type of mixed nonlinear PDEs of other orders, let it then be as

$$L\big(u(x,t)\big) + R\big(u(x,t)\big) + N\big(u(x,t)\big) = h(x,t)$$

$$\frac{\partial^n u}{\partial x \partial y^{n-1}} + R\big(u(x,t)\big) + N\big(u(x,t)\big) = h(x,t)$$

With U $=\frac{\partial^{n-1}u}{\partial y^{n-1}}$

$$\frac{\partial}{\partial x}\left(\frac{\partial^{n-1}u}{\partial y^{n-1}}\right) + R\big(u(x,t)\big) + N\big(u(x,t)\big) = h(x,t)$$

$$\frac{\partial U}{\partial x} + R\big(u(x,t)\big) + N\big(u(x,t)\big) = h(x,t)$$

$$u(x,0) = f_0(x)\,\frac{\partial u(0,y)}{\partial y^{n-1}} = g(y)\,\frac{\partial u(x,0)}{\partial y} = f_1(x)\,\frac{\partial^2 u(x,0)}{\partial y^2} = f_2(x)\ \ldots\ \frac{\partial^{n-2}u(x,0)}{\partial y^{n-2}} = f_{n-2}(x)$$

As can be seen, what needs to be generalized is thenth derivative of the ZJ Transformation, then, equivalently, y = t

$$ZJ\left[\frac{\partial^m u(x,t)}{\partial y^m}\right] = \left(\frac{z}{\beta}\right)^m \frac{\beta^n}{z}\int_0^\infty e^{-\frac{zt}{\beta}}u(x,t)dt - \frac{\beta^n}{z}\sum_{k=0}^{m-1}\left[\left(\frac{z}{\beta}\right)^{m-1-k}\right]\frac{\partial^k u(x,0)}{\partial t^k}$$

$$\hat{\varphi} = \frac{\beta^n}{z}\int_0^\infty e^{-\frac{zt}{\beta}}u(x,t)dt$$

Now, the development of some derivatives

Derivative	**Expression**
$ZJ\left[\frac{\partial u(x,t)}{\partial y}\right]$	$\frac{z}{\beta}\hat{\varphi} - \frac{\beta^n}{z}u(x,0)$
$ZJ\left[\frac{\partial^2 u(x,t)}{\partial y^2}\right]$	$\frac{z^2}{\beta^2}\hat{\varphi} - \frac{\beta^n}{\beta}u(x,0) - \frac{\beta^n}{z}u'(x,0)$
$ZJ\left[\frac{\partial^3 u(x,t)}{\partial y^3}\right]$	$\frac{z^3}{\beta^3}\hat{\varphi} - \frac{z\beta^n}{\beta^2}u(x,0) - \frac{\beta^n}{\beta}u'(x,0) - \frac{\beta^n}{z}u''(x,0)$
$ZJ\left[\frac{\partial^4 u(x,t)}{\partial y^4}\right]$	$\frac{z^4}{\beta^4}\hat{\varphi} - \frac{z^2\beta^n}{\beta^3}u(x,0) - \frac{z\beta^n}{\beta^2}u'(x,0) - \frac{\beta^n}{\beta}u''(x,0) - \frac{\beta^n}{z}u'''(x,0)$

We apply the ZJ transform on x, denoted by ZJ { · } , to both sides of the PDE:

ZJ { $\frac{\partial U}{\partial x}$} + ZJ { R (u (x, t))} + ZJ { N (u (x, t))} = ZJ { h (x, t)}

Using the properties of the ZJ transform for nth-order derivatives:

Step 2: Solution in the ZJ Domain by accommodating terms within the PDE

$$\widehat{U(x,y)} = \frac{\beta^{n+1}}{z^2}\,U(0,y) - \frac{\beta}{z}ZJ_x[Ru] - \frac{\beta}{z}ZJ_x[Nu] + \frac{\beta}{z}ZJ_x[h]$$

Replacing the boundary condition

$$\widehat{U(x,y)} = \frac{\beta^{n+1}}{z^2}\,g(y) - \frac{\beta}{z}ZJ_x[Ru] - \frac{\beta}{z}ZJ_x[Nu] + \frac{\beta}{z}ZJ_x[h]$$

Now taking the inverse ZJ transform with respect to x

$$\boldsymbol{U(x,y) = ZJ_x^{-1}}\left[\frac{\beta^{n+1}}{z^2}\,g(y)\right] + \boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[h - Ru - Nu]\right]$$

$$\frac{\partial^{n-1}u(x,y)}{\partial y^{n-1}} = g(y) + \boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[h - Ru - Nu]\right]$$

Now we have a nonlinear, nonhomogeneous, first-order PDE, taking the ZJ transform with respect to y

$$\boldsymbol{ZJ_y}\left[\frac{\partial^{n-1}u(x,y)}{\partial y^{n-1}}\right] = \boldsymbol{ZJ_y[g(y)]} + \boldsymbol{ZJ_y}\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\boldsymbol{\beta}}{\boldsymbol{z}}\boldsymbol{ZJ_x[h - Ru - Nu]}\right]\right]$$

Now this is the transform of thenth derivative, resulting in:

$$\widehat{u(x,y)} = \left(\frac{\beta}{z}\right)^m\left[\frac{\beta^n}{z}\sum_{k=0}^{m-1}\left[\left(\frac{z}{\beta}\right)^{m-1-k}\right]\frac{\partial^k u(x,0)}{\partial t^k}\right] + \left(\frac{\beta}{z}\right)^m ZJ_y[g(y)] + \left(\frac{\beta}{z}\right)^m ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[h - Ru - Nu]\right]\right]$$

Now taking the inverse ZJ transform with respect to y and

$$\boldsymbol{u(x,y)} = \boldsymbol{ZJ_y^{-1}}\left[\left(\frac{\beta}{z}\right)^m\left[\frac{\beta^n}{z}\sum_{k=0}^{m-1}\left[\left(\frac{z}{\beta}\right)^{m-1-k}\right]\frac{\partial^k u(x,0)}{\partial t^k}\right]\right] + \boldsymbol{ZJ_y^{-1}}\left[\left(\frac{\beta}{z}\right)^m ZJ_y[g(y)]\right] + \boldsymbol{ZJ_y^{-1}}\left[\left(\frac{\beta}{z}\right)^m ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[h - Ru - Nu]\right]\right]\right]$$

The indicated transformation expression is obtained.

$$\boldsymbol{u(x,y)} = \boldsymbol{\alpha(x,y)} + \boldsymbol{ZJ_y^{-1}}\left[\left(\frac{\beta}{z}\right)^m ZJ_y[g(y)]\right] + \boldsymbol{ZJ_y^{-1}}\left[\left(\frac{\beta}{z}\right)^m ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[h - Ru - Nu]\right]\right]\right]$$

Step 3: Application of the Adomian Decomposition Method Finally, the general recursive system remains.

$$u_0 = \boldsymbol{\alpha(x,y)} + \boldsymbol{ZJ_y^{-1}}\left[\left(\frac{\beta}{z}\right)^m ZJ_y[g(y)]\right] + \boldsymbol{ZJ_y^{-1}}\left[\left(\frac{\beta}{z}\right)^m ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x[h]\right]\right]\right]$$

$$\sum_{\boldsymbol{n=0}}^{\infty}\boldsymbol{u_{n+1}} = -\boldsymbol{ZJ_y^{-1}}\left[\left(\frac{\beta}{z}\right)^m ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\frac{\beta}{z}ZJ_x\left[R\sum_{n=0}^{\infty}u_n(x,y) + \sum_{n=0}^{\infty}A_n(x,y)\right]\right]\right]\right]$$

Let's look at an example

$$\frac{\partial^3 \boldsymbol{u}}{\partial \boldsymbol{x}^2\partial \boldsymbol{y}} - \boldsymbol{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{y}}\right)^3 = \boldsymbol{0}$$

With $\boldsymbol{u(x,0) = 0}$ $\quad \frac{\partial u(0,y)}{\partial y} = 1 \quad \frac{\partial u(0,y)}{\partial xy} = 1$

We apply the ZJ transform on x, denoted by ZJ { ·} , to both sides of the PDE:

ZJ { $\frac{\partial^2 U}{\partial x^2}$} -2 ZJ { U^3} = 0

Using the properties of the ZJ transform for nth-order derivatives:

Step 2: Solution in the ZJ Domain by accommodating terms within the PDE

$$\widehat{U(x,y)} = \frac{\beta^{n+2}}{z^3} + \frac{\beta^{n+1}}{z^2} + 2\frac{\beta^2}{z^2}ZJ_x[U^3]$$

Now taking the inverse ZJ transform with respect to ax

$$\boldsymbol{U(x,y)} = \boldsymbol{ZJ_x^{-1}}\left[\frac{\beta^{n+2}}{z^3}\right] + \boldsymbol{ZJ_x^{-1}}\left[\frac{\beta^{n+1}}{z^2}\right] + \boldsymbol{2}\frac{\beta^2}{z^2}\boldsymbol{ZJ_x^{-1}}[ZJ_x[U^3]]$$

$$\frac{\partial u}{\partial y} = x + 1 + \boldsymbol{ZJ_x^{-1}}\left[\boldsymbol{2}\frac{\beta^2}{z^2}ZJ_x[U^3]\right]$$

Now we have a nonlinear, nonhomogeneous, first-order PDE, taking the ZJ transform with respect to y, and considering the following condition u(x,0) = 0

$$\boldsymbol{ZJ_y}\left[\frac{\partial u}{\partial y}\right] = \boldsymbol{ZJ_y[x]} + \boldsymbol{ZJ_y[1]} + \boldsymbol{ZJ_y}\left[\boldsymbol{ZJ_x^{-1}}\left[\boldsymbol{2}\frac{\beta^2}{z^2}ZJ_x[U^3]\right]\right]$$

Now taking the inverse ZJ transform with respect to y, simplifying

$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{xy} + \boldsymbol{y} + \boldsymbol{ZJ_y^{-1}}\left[\frac{\beta}{z}ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\boldsymbol{2}\frac{\beta^2}{z^2}ZJ_x[U^3]\right]\right]\right]$$

Now the system is considered as

$$u_0 = \boldsymbol{xy} + \boldsymbol{y}$$

$$\sum_{\boldsymbol{n=0}}^{\infty}\boldsymbol{u_{n+1}} = \boldsymbol{ZJ_y^{-1}}\left[\frac{\beta}{z}ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\boldsymbol{2}\frac{\beta^2}{z^2}ZJ_x[U^3]\right]\right]\right]$$

By the Adomian method

$$\boldsymbol{Nu} = \left(\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial y}}\right)^{\boldsymbol{3}} = \sum_{\boldsymbol{n=0}}^{\infty}\boldsymbol{A_n}$$

Having similar Adomian polynomials

$$A_0 = u_0^3$$
$$A_1 = 3u_0^2u_1$$
$$A_2 = 3u_0u_1^2 + 3u_0^2u_2$$

With

$$\left(\frac{\boldsymbol{\partial u_0}}{\boldsymbol{\partial y}}\right)^{\boldsymbol{3}} = A_0 = (1+x)^3$$

Thus with n = 0

$$\sum_{\boldsymbol{n=0}}^{\infty}\boldsymbol{u_1} = \boldsymbol{ZJ_y^{-1}}\left[\frac{\beta}{z}ZJ_y\left[\boldsymbol{ZJ_x^{-1}}\left[\boldsymbol{2}\frac{\beta^2}{z^2}ZJ_x[(1+x)^3]\right]\right]\right]$$

Solving and simplifying, we finally have

$$u_1 = 3x^2y + x^3y + \frac{x^4y}{2} + \frac{x^5y}{10}$$

Therefore, when forming the series we have

$$u_n = u_0 + u_1 + u_2 + \cdots$$
$$u_n = y + xy + 3x^2y + x^3y + \frac{x^4y}{2} + \frac{x^5y}{10} + \cdots$$

By combining the other very long terminus, the type of series that converges is observed; finally, we have the analytical solution.

$$u(x,y)_n = y\sum_{n=0}^{\infty}x^n = \frac{y}{1-x}$$

Conclusions

The combination of the Adomian Decomposition Method (ADM) with the ZJ transform (Adomian-ZJ method), as well as other Adomian-Laplace methods, offers several significant advantages over classical methods for solving certain nonlinear PDEs:

However, it is important to take into account some limitations, such as in the case of nonlinear ODEs:

- The convergence of the Adomian series is not always guaranteed and may depend on the nature of the nonlinear equation and the conditions of the problem.
- Calculating Adomian polynomials for complex nonlinearities can be laborious.
- Applying the inverse ZJ transform to the components $\hat{\varphi}_n(x,t)$may not always be trivial and may require special techniques or the use of transform tables.

The Adomian-ZJ method offers a powerful and elegant alternative to classical methods for solving certain nonlinear PDEs, especially those where linearization or perturbations are not suitable or where an approximate an alytical solution is sought without resorting to discretization.

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References

About the Adomian Decomposition Method (ADM):

[1]. Adomian, G. (1988). *Nonlinear stochastic operator equations*. Academic Press .
[2]. Adomian, G. (1994). *Solving nonlinear partial differential equations with Adomian's decomposition method*. Journal of Mathematics Analysis and Applications, *186* (3), 802-814.
[3]. Cherruault, Y. (1989). *Convergence of Adomian's method*. Kybernetes, *18* (2), 31-38.
[4]. Datta, Mousumi & Habiba, Umme & Hossain, (2020). Elzaki Substitution Method for Solving Nonlinear Partial Differential Equations with Mixed Partial Derivatives Using A domain Polynomial. 10.12691/ijpdea-8-1-2.
[5]. Datta , Mousumi & Hossain, Md. Babul & Habiba , Umme . (2022). Elzaki Substitution Method with Adomian Polynomials for Solving Initial Value Problems of n th Order Non-linear Mixed Type Partial Differential Equations. 649-662.
[6]. Wazwaz, A. M. (2009). *The Adomian decomposition method for nonlinear integral equations*. Applied Mathematics and Computation, *207* (1-2), 228-234.
[7]. Rach, R. (1984). *On the Adomian decomposition method and some related topics*. Journal of MathematicsAnalysis and Applications, *99* (2), 431-450.
[8]. Zenteno Jiménez José RobertoInternational Journal of Latest Research in Engineering and Technology (IJLRET) ISSN: 2454-5031 www.ijlret.com ‖ Volume 11 - Issue 07 ‖ July 2025 ‖ PP. 17-26 A Solution to Nonlinear Ordinary Differential Equations using the Adomian Decomposition Method and the ZJ Transform
[9]. Zenteno Jimenez Jose Roberto, International Journal of Latest Research in Engineering and Technology (IJLRET) ISSN: 2454-5031 www.ijlret.com ‖ Volume 11 - Issue 01 ‖ January 2025 ‖ PP. 01-10 Summary of the new ZJ Transform and its application to Differential Equations, Partial Equations and Integral Equations
[10]. Zenteno Jiménez José Roberto, International Journal of Latest Research in Engineering and Technology (IJLRET) ISSN: 2454-5031 www.ijlret.com ‖ Volume 09 - Issue 02 ‖ February 2023 ‖ PP. 07-12 Applications of the ZJ Transform for Differential Equations On Combining the Adomian Method with the Laplace Transform (LADM):
[11]. Khuri, S. A. (2001). *The Laplace decomposition method for solving nonlinear differential equations of second order*. Applied Mathematics and Computation, *124* (1), 147-155.
[12]. Yildirim, A. (2007). *Application of Laplace transform decomposition method to nonlinear differential equations.* Applied Mathematics and Computation, *186* (2), 930-934.