International Journal of Latest Research in Engineering and Technology (IJLRET)
ISSN: 2454-5031
www.ijlret.com || Volume 12 - Issue 01 || January 2026 || PP. 34-53

New Representation of Integer Numbers Based on 2's-Powered, Their Algebraic Relations and Their Applications (Fermat's Last Theorem)

Hossain Ghaffari (Misragoras)[a], Abbas Ghaffari[b*]
[a]*Department of Philosophy, Tabriz University, Tabriz, Iran*
[b]*Department of Mathematics, Science Research, Islamic Azad University, Tehran, Iran*

Abstract: We present a binary-power decomposition for every integer; express an integer a as the sum of successive powers of 2 and identify any missing terms (called defectors). This yields a unique partition of a into a complete part (a full geometric sequence of 2-powers) minus a defector sum. We introduce the notion of an integer's rank (the highest exponent in its decomposition) and develop an algebra governing parity and exponentiation directly from these binary-power sums. As a principal application, we revisit the Diophantine equation $(x^n + y^n = z^n \, for \, n \geq 3)$. By analyzing parity patterns and exploiting uniqueness of the binary-power decomposition, we show that no nontrivial solutions exist in $\mathbb{N}^3$ when $n \geq 3$, recovering Fermat's Last Theorem through an elementary combinatorial-parity argument.

Keywords: binary-power decomposition, complete integer, incomplete integer, defector, Fermat's Last Theorem, Diophantine equations.

1. Introduction

The binary expansion of integers underlies both computer science and number theory, yet its direct role in structural Diophantine arguments remains underutilized. In this paper, we develop a systematic algorithm to decompose any integer $\boldsymbol{a}$ intoodd number= $\boldsymbol{a} = (\boldsymbol{2^m} + \boldsymbol{2^{m-1}} + \cdots + \boldsymbol{2^1} + \boldsymbol{1}) - (\boldsymbol{2^{m-(k_1-1)}} + \boldsymbol{2^{m-(k_2-1)}} + \cdots + \boldsymbol{2^{m-(k_r-1)}})$ "complete odd numbers-defectors" or even number $= \boldsymbol{b} = \left(\boldsymbol{2^{m'}} + \boldsymbol{2^{m'-1}} + \cdots + \boldsymbol{2^1}\right) - (\boldsymbol{2^{m'-(k'_1-1)}} + \cdots + \boldsymbol{2^{m'-(k'_{r'}-1)}})$"complete even numbers - defectors", where "Defectors" are omitted 2-powers that prevent the sum from being a full geometric progression. By classifying integers as complete (no defectors) or incomplete (one or more defectors), we obtain: A unique representation for each integer. A natural rank function $\boldsymbol{r(a)}$ = highest exponent in the full-sum model.

A concise algebra of parity, tracking how sums and $\boldsymbol{n}$th-powers move between odd numbers and even numbers forms. We then apply these tools to the classical Fermat equation $(\boldsymbol{x^n + y^n = z^n \, for \, n \geq 3})$ Parity constraints force the only viable pattern ***"odd number + odd number = even number"***. Combined with the uniqueness of the binary-power decomposition, this parity analysis suffices to exclude all nontrivial solutions for $\boldsymbol{n \geq 3}$. Our method revisits Fermat's Last Theorem from a fresh, purely arithmetic perspective.

2. New representation of integer numbers based on 2's-powered and their algebraic relations

2.1. preliminaries

In following,every number in $\mathbb{N}$ or $\mathbb{Z}$ can be show in various representationsuch as:

(i) $\boldsymbol{a = p_1^{\alpha_1}.p_2^{\alpha_2} \ldots p_r^{\alpha_r}, r \in \mathbb{N}, p_i \in \mathbb{Z}, a_i \in \mathbb{N} \; and \; p_i \; is \; prime \; number}$;

(ii) $\boldsymbol{a = 2^{r_1}.a_1 + 1 \; or \; a = 2^{r_2}.a_2, a_i \in \mathbb{Z}, i = 1,2 \;,\; r_1, r_2 \in \mathbb{N}}$.

From the point of view of properties, there is twokindof numbers (in $\mathbb{Z}$), as follows:

(1) odd numbers, we show generally with $\boldsymbol{a = 2k + 1}$, $\boldsymbol{k} \in \mathbb{Z}$ and we will represent it by "$\boldsymbol{i}$";

(2) even numbers, we show generally with $\boldsymbol{a = 2k}$, $\boldsymbol{k} \in \mathbb{Z}$ and we will represent it by "$\boldsymbol{p}$".

Today in mathematics, there is a current representation of numbers, related to be odd or even forms that they are shown in (i),(ii). We notice to numbers from the point of 2's-powered, and we are getting their properties upon this representation.

Definition 2.1.1.Algorithm of representing of integer numbers, based on 2- powered.

Suppose$\boldsymbol{i = a = 2^r.k_1 + 1}$, $\boldsymbol{r} \in \mathbb{N}, \boldsymbol{k_1} \in \mathbb{Z}$, then $\boldsymbol{k_1}$ has the two below cases upon the **system** $\boldsymbol{i - p}$ **,** $(\boldsymbol{S_{i-p}})$**:**

(1) $\boldsymbol{k_1 = 2.k_2 + 1}$, $\boldsymbol{k_2} \in \mathbb{Z}$, so

$$\boldsymbol{i = a = 2^r.k_1 + 1}$$
$$\boldsymbol{= 2^r.(2.k_2 + 1) + 1}$$

$$= \mathbf{2^{r+1}.k_2 + 2^r + 1}, \qquad (\mathbf{2.1})$$

(2) $\mathbf{k_1 = 2k_2}$, $\mathbf{k_2} \in \mathbb{Z}$, so

$$\begin{aligned} \boldsymbol{i = a} &= \mathbf{2^r.k_1 + 1} \\ &= \mathbf{2^r.(2.k_2) + 1} \\ &= \mathbf{2^{r+1}.k_2 + 1}, \qquad (\mathbf{2.2}) \end{aligned}$$

(1-1) $\mathbf{k_2 = 2.k_3 + 1}$, $\mathbf{k_3} \in \mathbb{Z}$, so, (2-1) become as follows

$$\begin{aligned} \boldsymbol{i = a} &= \mathbf{2^{r+1}.k_2 + 2^r + 1} \\ &= \mathbf{2^{r+1}.(2.k_3 + 1) + 2^r + 1} \\ &= \mathbf{2^{r+2}.k_3 + 2^{r+1} + 2^r + 1}, \end{aligned}$$

(1-2) $\mathbf{k_2 = 2k_3}$, $\mathbf{k_3} \in \mathbb{Z}$, so, (2-1) become as follows

$$\begin{aligned} \boldsymbol{i = a} &= \mathbf{2^{r+1}.k_2 + 2^r + 1} \\ &= \mathbf{2^{r+1}.(2.k_3) + 2^r + 1} \\ &= \mathbf{2^{r+2}.k_3 + 2^r + 1}, \end{aligned}$$

(2-1) $\mathbf{k_2 = 2.k_3 + 1}$, $\mathbf{k_3} \in \mathbb{Z}$, so, (2-2) become as follows

$$\begin{aligned} \boldsymbol{i = a} &= \mathbf{2^{r+1}.k_2 + 1} \\ &= \mathbf{2^{r+1}.(2.k_3 + 1) + 1} \\ &= \mathbf{2^{r+2}.k_3 + 2^{r+1} + 1}, \end{aligned}$$

(2-2) $\mathbf{k_2 = 2.k_3}$, $\mathbf{k_3} \in \mathbb{Z}$, so, (2-2) become as follows

$$\begin{aligned} \boldsymbol{i = a} &= \mathbf{2^{r+1}.k_2 + 1} \\ &= \mathbf{2^{r+1}.(2.k_3) + 1} \\ &= \mathbf{2^{r+2}.k_3 + 1}. \end{aligned}$$

By continuing this process, we will have the following extensions of 2's-powered,

$\boldsymbol{i = a = 2^\alpha + 2^\beta + \cdots + 2^\theta + 2^\Omega}$, $\boldsymbol{\alpha > \beta > \cdots > \theta > \Omega \geq 0}$,

similarly, if $\boldsymbol{a = p = 2^r.k_1}$, $\boldsymbol{r} \in \mathbb{N}$, $\boldsymbol{k_1} \in \mathbb{Z}$ then we will have the following extensions of 2's-powered,

$\boldsymbol{p = a = 2^\alpha + 2^\beta + \cdots + 2^\theta + 2^\Omega}$, $\boldsymbol{\alpha > \beta > \cdots > \theta > \Omega} > 0$.

2.2. Representation of integer numbers based on 2's-powered and their algebraic relations.

Definition 2.2.1. Let $\boldsymbol{a} \in \mathbb{Z}$and $\boldsymbol{a = 2^\alpha + 2^\beta + \cdots + 2^\theta + 2^\Omega}$, $\boldsymbol{\alpha > \beta > \cdots > \theta > \Omega \geq 0}$,if sentences of number $\mathbf{2^\alpha + 2^\beta + \cdots + 2^\theta + 2^\Omega}$ are sentences of a geometrical progression with respect value of $\mathbf{2}$, then "$\boldsymbol{a}$" is a complete number and otherwise "$\boldsymbol{a}$" is incomplete number.

Example 2.2.2. If $\boldsymbol{i = a = 2^m + 2^{m-1} + \cdots + 2^1 + 1}$ then finite sequence $(\mathbf{1, 2^1, \ldots, 2^{m-1}, 2^m})$ is a geometrical progression with respect value of 2. Therefore $\boldsymbol{i = a}$ is an odd complete number.

Example 2.2.3. If $\boldsymbol{p = b = 2^{m'} + 2^{m'-1} + \cdots + 2^3 + 2}$ then finite sequence $(\mathbf{2, 2^3, \ldots, 2^{m'-1}, 2^{m'}})$ is not a geometrical progression because sequence $(\mathbf{2, 2^3, \ldots, 2^{m'-1}, 2^{m'}})$has a defector "$\mathbf{2^2}$". Therefore $\boldsymbol{p = b}$ is an even incomplete number and "$\mathbf{2^2}$" is a defector.

Definition 2.2.4. Base on the representation of 2's-powered, every integer number is aomplete or incomplete number. We present every incomplete number to two parts, at first, we have complete part and in second part, we have the defector part so,

incomplete number = complete part − defector part

the complete part is sum of the sentences of a geometrical progression with respect value 2 and defector part is sum of sentences which their absence causes incompleteness of that number. Sentences of defector maybe there in one or more point of complete part such as successive or periodical.Therefore general presentation of integer number based on 2's-powered may be shown as the following forms:

If $\boldsymbol{a = i}$, then

$$\begin{aligned} \boldsymbol{i = a} &= (\mathbf{2^m + 2^{m-1} + \cdots + 2^1 + 1}) - (\mathbf{2^{m-(k_1-1)} + 2^{m-(k_2-1)} + \cdots + 2^{m-(k_r-1)}}) \\ &= (\sum_{\boldsymbol{j=0}}^{\boldsymbol{m}} \mathbf{2^j}) - (\mathbf{2^{m-(k_1-1)} + 2^{m-(k_2-1)} + \cdots + 2^{m-(k_r-1)}}) \\ &= (\mathbf{2^{m+1} - 1}) - \left(\mathbf{2^{m-(k_1-1)} + 2^{m-(k_2-1)} + \cdots + 2^{m-(k_r-1)}}\right), \qquad (\mathbf{2.3}) \end{aligned}$$

and if $\boldsymbol{b = p}$, then

$$p = b = \left(2^{m'} + 2^{m'-1} + \cdots + 2^1\right) - (2^{m'-(k'_1-1)} + \cdots + 2^{m'-(k'_{r'}-1)})$$
$$= (\sum_{j=1}^{m'} 2^j) - (2^{m'-(k'_1-1)} + \cdots + 2^{m'-(k'_{r'}-1)})$$
$$= \left(2^{m'+1} - 2\right) - \left(2^{m'-(k'_1-1)} + \cdots + 2^{m'-(k'_{r'}-1)}\right). \qquad (2.4)$$

Notice. If $i = a = (2^{m+1} - 1) - \left(2^{m-(k_1-1)} + 2^{m-(k_2-1)} + \cdots + 2^{m-(k_r-1)}\right)$ then two part is as follows,
(i) complete part $= 2^m + 2^{m-1} + \cdots + 2^1 + 1$
$= 2^{m+1} - 1$;
(ii) defector part $= 2^{m-(k_1-1)} + 2^{m-(k_2-1)} + \cdots + 2^{m-(k_r-1)}$.
Also, if $p = b = \left(2^{m'+1} - 2\right) - \left(2^{m'-(k'_1-1)} + \cdots + 2^{m'-(k'_{r'}-1)}\right)$then two part is as follows,
(j) complete part$= 2^{m'} + 2^{m'-1} + \cdots + 2^1$
$= 2^{m'+1} - 2$;
(jj) defector part $= 2^{m'-(k'_1-1)} + \cdots + 2^{m'-(k'_{r'}-1)}$.
If $a = i$ then the range of changes for defector part is from 0 to all 2's-powered of geometrical progression, expect first sentence and last sentence. It means that, the least amount of defector of number,
$i = a = (2^{m+1} - 1) - \left(2^{m-(k_1-1)} + 2^{m-(k_2-1)} + \cdots + 2^{m-(k_r-1)}\right)$ is as follows,
$$i = a = (2^{m+1} - 1)$$
$$= 2^m + 2^{m-1} + \cdots + 2^1 + 1 ,$$
and the most amount of defector of number,
$i = a = (2^{m+1} - 1) - \left(2^{m-(k_1-1)} + 2^{m-(k_2-1)} + \cdots + 2^{m-(k_r-1)}\right)$ is as follows,
$$i = a = (2^m + 2^{m-1} + \cdots + 2^1 + 1) - (2^{m-1} + \cdots + 2^1)$$
$$= 2^m + 1.$$
Similarly, the least amount of defector of number$p = b$,
$p = b = \left(2^{m'+1} - 2\right) - \left(2^{m'-(k'_1-1)} + \cdots + 2^{m'-(k'_{r'}-1)}\right)$is as follows,
$$p = b = \left(2^{m'+1} - 2\right)$$
$$= 2^{m'} + 2^{m'-1} + \cdots + 2^1,$$
and the most amount of defector of number,
$p = b = \left(2^{m'+1} - 2\right) - \left(2^{m'-(k'_1-1)} + \cdots + 2^{m'-(k'_{r'}-1)}\right)$ is as follows,
$$p = b = \left(2^{m'} + 2^{m'-1} + \cdots + 2^1\right) - (2^{m'-1} + \cdots + 2^1)$$
$$= 2^{m'}.$$
Notice. According to algorithm of representing of integer numbers based on 2's-powered, representation of $a = 2^\alpha + 2^\beta + \cdots + 2^\theta + 2^\Omega$, $\alpha > \beta > \cdots > \theta > \Omega \geq 0$ is unique.Alsoit will be proved later in the form of a theorem. Therefore 2's-powered are recognizer of identification of the number a, whether it is complete or incomplete.

Definition 2.2.5. If $a = 2^\alpha + 2^\beta + \cdots + 2^\theta + 2^\Omega$, $\alpha > \beta > \cdots > \theta > \Omega \geq 0$ we say rank "a" is equal to greatest power of the representationof "a" and we show it by$\lambda(a) = \alpha$.Also we say two integer numbers a, b are homorank if and only if $\lambda(a) = \lambda(b)$, whether they both are complete or incomplete, or a is complete and b is incomplete or vise versa.

Theorem 2.2.6. Inherence of homorank for an equality.
Let $a, b \in \mathbb{Z}$ and $a = b$ then, $\lambda(a) = \lambda(b)$.
We consider
$$a = 2^\alpha + 2^\beta + \cdots + 2^\theta + 2^\Omega , \quad \alpha > \beta > \cdots > \theta > \Omega \geq 0.$$
$$b = 2^{\alpha'} + 2^{\beta'} + \cdots + 2^{\theta'} + 2^{\Omega'} , \quad \alpha' > \beta' > \cdots > \theta' > \Omega' \geq 0.$$
from $a = b$, we have
$$2^\alpha + 2^\beta + \cdots + 2^\theta + 2^\Omega = 2^{\alpha'} + 2^{\beta'} + \cdots + 2^{\theta'} + 2^{\Omega'} \qquad (2.5)$$
if there exist $\omega \in \mathbb{N}$ such that $2^\omega$is sentence of a and 2^ω isn't sentences of b , by factorization 2^ω from (2.5)

we have

$2^{\alpha} + 2^{\beta} + \dots + 2^{\omega+t} + 2^{\omega} + 2^{\omega-l} + \dots + 2^{\theta} + 2^{\Omega} = 2^{\alpha'} + 2^{\beta'} + \dots + 2^{\theta'} + 2^{\Omega'}$ so

$$2^{\alpha-\omega} + 2^{\beta-\omega} + \dots + 2^{t} + 2^{0} + 2^{-l} + \dots + 2^{\theta-\omega} + 2^{\Omega-\omega} = 2^{\alpha'-\omega} + 2^{\beta'-\omega} + \dots + 2^{\theta'-\omega} + 2^{\Omega'-\omega}, \quad (2.6)$$

from (2.6) we have

$$2^{\alpha-\omega} + 2^{\beta-\omega} + \dots + 2^{t} + 2^{0} = i$$
$$2^{-l} + \dots + 2^{\theta-\omega} + 2^{\Omega-\omega} = fractional\ number$$
$$2^{\alpha'-\omega} + 2^{\beta'-\omega} + \dots + 2^{\theta'-\omega} + 2^{\Omega'-\omega} = p + fractional\ number,$$

hence, in S_{i-p} , (2.6) is become as follows,

$$2^{\alpha-\omega} + \dots + 2^{t} + 2^{0} + 2^{-l} + \dots + 2^{\theta-\omega} + 2^{\Omega-\omega} = 2^{\alpha'-\omega} + \dots + 2^{\theta'-\omega} + 2^{\Omega'-\omega}$$

$i + fractional\ number = p + fractional\ number$ therefore

$i = i - p = fractional\ number,$

it is contradiction. ■

Corollary 2.2.7. According to theorem **2.2.6** an integer number cannot be coincide complete and incomplete.

Theorem 2.2.8. (The greatest of defector number $a = i$) If $a = i$ is incomplete number and it has greatest of defector, then,$a - incomplete\ number's\ part = 3$.

Proof. Let $= a = (2^{m} + 2^{m-1} + \dots + 2^{1} + 1) - (2^{m-(k_1-1)} + + \dots + 2^{m-(k_r-1)})$,

we put $a_1 = 2^{m} + 2^{m-1} + \dots + 2^{1} + 1$ and $a_2 = 2^{m-(k_1-1)} + + \dots + 2^{m-(k_r-1)}$ so $a = a_1 - a_2$. Since it may not be there 2^0=1 or 2^m in $a_2$because if there is 1 ina_2 then $a = p$ and it is contradiction and if there is 2^m in a_2 then $\lambda(a) > \lambda(a)$, it is contradiction. Therefore when

$i = a = (2^{m} + 2^{m-1} + \dots + 2^{1} + 1) - (2^{m-1} + \dots + 2^{1})$ we have $a = 2^{m} + 1$ and

$$\begin{aligned} a_2 &= 2^{m-1} + \dots + 2^{1} \\ &= 2(2^{m-2} + \dots + 1) \\ &= 2(2^{m-1} - 1) \\ &= 2^{m} - 2 \\ &= (2^{m} + 1) - 3 \\ &= a - 3 \end{aligned}$$

Hence $a_2 = a - 3$ and $a_2 < a$. ■

Theorem 2.2.9. Let $a = i$ be an incomplete number and $c = i$ is a complete number also $\lambda(a) = m$ and $\lambda(c) = m'$ then, $\lambda(a + c) = max\{\lambda(a), \lambda(c)\} + 1$ or $\lambda(a + c) = max\{\lambda(a), \lambda(c)\}$

Proof. If $a = i = (2^{m} + 2^{m-1} + \dots + 2^{1} + 1) - (2^{m-(k_1-1)} + \dots + 2^{m-(k_r-1)})$

and $c = (2^{m'+1} - 1)$ then $\lambda(a) = m$ and $\lambda(c) = m'$, we have three cases between m and m',

(1) $m = m'$;

(2) $m > m'$;

(3) $m < m'$;

(1) If $m = m'$ then,

$$\begin{aligned} a + c &= [(2^{m+1} - 1) - (2^{m-(k_1-1)} + \dots + 2^{m-(k_r-1)})] + (2^{m+1} - 1) \\ &= (2^{m+2} - 2) - (2^{m-(k_1-1)} + \dots + 2^{m-(k_r-1)}) \\ &= (2^{m+1} + 2^{m} + \dots + 2) - (2^{m-(k_1-1)} + \dots + 2^{m-(k_r-1)}), \end{aligned} \quad (2.7)$$

from (2.7) we have $\lambda(a + c) = m + 1$.

(2) If $m > m'$ then,

$$\begin{aligned} a + c &= [(2^{m+1} - 1) - (2^{m-(k_1-1)} + \dots + 2^{m-(k_r-1)})] + (2^{m'+1} - 1) \\ &= [(2^{m+1} - 2) + 2^{m'+1} - (2^{m-(k_1-1)} + \dots + 2^{m-(k_r-1)})] \\ &= [(2^{m} + 2^{m-1} + \dots + 2) + 2^{m'+1} - (2^{m-(k_1-1)} + \dots + 2^{m-(k_r-1)})] \\ &= [(2^{m} + 2^{m-1} + \dots + 2^{m'+1} + 2^{m'} + \dots + 2) + 2^{m'+1} - (2^{m-(k_1-1)} + \dots + 2^{m-(k_r-1)})] \\ &= (2^{m+1} + 2^{m'} + \dots + 2) - (2^{m-(k_1-1)} + \dots + 2^{m-(k_r-1)}) \\ &= (2^{m+1} + 2^{m} + \dots + 2^{m'} + \dots + 2) - (2^{m} + \dots + 2^{m'+1}) - (2^{m-(k_1-1)} + \dots + 2^{m-(k_r-1)}) \\ &= (2^{m+1} + 2^{m} + \dots + 2^{m'} + \dots + 2) - (2^{m} + \dots + 2^{m'+1} + 2^{m-(k_1-1)} + \dots + 2^{m-(k_r-1)}), \end{aligned}$$

if $\boldsymbol{m}' < m-(\boldsymbol{k_1}-\mathbf{1})$ then $\boldsymbol{\lambda}(\boldsymbol{a}+\boldsymbol{c})=\boldsymbol{\lambda}(\boldsymbol{a})=\boldsymbol{m}$ and if $\boldsymbol{m}' = \boldsymbol{m}-(\boldsymbol{k_1}-\mathbf{1})$ then $\boldsymbol{\lambda}(\boldsymbol{a}+\boldsymbol{c})=\boldsymbol{\lambda}(\boldsymbol{a})+\mathbf{1}=\boldsymbol{m}+\mathbf{1}$ and if $\boldsymbol{m}' > m-(\boldsymbol{k_1}-\mathbf{1})$ then $\boldsymbol{\lambda}(\boldsymbol{a}+\boldsymbol{c})=\boldsymbol{\lambda}(\boldsymbol{a})+\mathbf{1}=\boldsymbol{m}+\mathbf{1}$.

(3) If $\boldsymbol{m}<\boldsymbol{m}'$ then,

$$\boldsymbol{a}+\mathbf{c}=[(\mathbf{2^{m+1}}-\mathbf{1})-(\mathbf{2^{m-(k_1-1)}}+\cdots+\mathbf{2^{m-(k_r-1)}})]+(\mathbf{2^{m'+1}}-\mathbf{1})$$
$$=[(\mathbf{2^{m'+1}}-\mathbf{2})+\mathbf{2^{m+1}}-(\mathbf{2^{m-(k_1-1)}}+\cdots+\mathbf{2^{m-(k_r-1)}})]$$
$$=[(\mathbf{2^{m'}}+\mathbf{2^{m'-1}}+\cdots+\mathbf{2})+\mathbf{2^{m+1}}-(\mathbf{2^{m-(k_1-1)}}+\cdots+\mathbf{2^{m-(k_r-1)}})]$$
$$=[(\mathbf{2^{m'}}+\mathbf{2^{m'-1}}+\cdots+\mathbf{2^{m+1}}+\mathbf{2^{m}}+\cdots+\mathbf{2})+\mathbf{2^{m+1}}-(\mathbf{2^{m-(k_1-1)}}+\cdots+\mathbf{2^{m-(k_r-1)}})]$$
$$=[(\mathbf{2^{m'+1}}+\mathbf{2^{m}}+\cdots+\mathbf{2})-(\mathbf{2^{m-(k_1-1)}}+\cdots+\mathbf{2^{m-(k_r-1)}})]$$
$$=[(\mathbf{2^{m'+1}}+\mathbf{2^{m'}}+\cdots+\mathbf{2^{m}}+\cdots+\mathbf{2})-(\mathbf{2^{m'}}+\cdots+\mathbf{2^{m+1}}+\mathbf{2^{m-(k_1-1)}}+\cdots+\mathbf{2^{m-(k_r-1)}})] \quad (2.8)$$

from$(\mathbf{2.8})$ we have $\boldsymbol{\lambda}(\boldsymbol{a}+\boldsymbol{c})=\boldsymbol{m}'+\mathbf{1}$.

Theorem 2.2.10. Let $\boldsymbol{a}$ and $\boldsymbol{c}$ be as follows

$$\boldsymbol{i}=\boldsymbol{a}=(\mathbf{2}^{\boldsymbol{m}+\mathbf{1}}-\mathbf{1})-(\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_1}-\mathbf{1})}+\cdots+\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_r}-\mathbf{1})})$$
$$\boldsymbol{i}=\boldsymbol{c}=(\mathbf{2}^{\boldsymbol{m}+\mathbf{1}}-\mathbf{1})+(\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_1}-\mathbf{1})}+\cdots+\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_r}-\mathbf{1})})$$

Then $\boldsymbol{\lambda}(\boldsymbol{c})=\boldsymbol{\lambda}(\boldsymbol{a})+\mathbf{1}$.

Proof. We have $\boldsymbol{i}=\boldsymbol{c}=(\mathbf{2^{m+1}}-\mathbf{1})+(\mathbf{2^{m-(k_1-1)}}+\cdots+\mathbf{2^{m-(k_r-1)}})$, if we choose $\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_r}-\mathbf{1})}$ from defector number part of $\boldsymbol{a}=\boldsymbol{i}$ since there exist $\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_r}-\mathbf{1})}$ in complete part of $\boldsymbol{c}=\boldsymbol{i}$ so we have

$$\boldsymbol{i}=\boldsymbol{c}=(\mathbf{2}^{\boldsymbol{m}+\mathbf{1}}-\mathbf{1})+(\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_1}-\mathbf{1})}+\cdots+\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_r}-\mathbf{1})})$$
$$=(\mathbf{2}^{\boldsymbol{m}}+\cdots+\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_1}-\mathbf{1})}+\cdots+\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_r}-\mathbf{1})}+\cdots+\mathbf{2}+\mathbf{1})(\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_1}-\mathbf{1})}+\cdots+\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_r}-\mathbf{1})})$$
$$=(\mathbf{2}^{\boldsymbol{m}+\mathbf{1}}+\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_r})}+\cdots+\mathbf{2}+\mathbf{1})+(\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_1}-\mathbf{1})}+\cdots+\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_{r-1}}-\mathbf{1})})$$
$$=(\mathbf{2}^{\boldsymbol{m}+\mathbf{1}}+\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_1}-\mathbf{1})}+\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_{r-1}}-\mathbf{1})}+\mathbf{2}^{\boldsymbol{m}-(\boldsymbol{k_r})}+\cdots+\mathbf{2}+\mathbf{1}). \quad (\mathbf{2.9})$$

From $(\mathbf{2.9})$, we conclude $\boldsymbol{\lambda}(\boldsymbol{c})=\boldsymbol{\lambda}(\boldsymbol{a})+\mathbf{1}=\boldsymbol{m}+\mathbf{1}$. ■

Theorem 2.2.11. If $\boldsymbol{a}=\boldsymbol{i}$ is a complete number and $\boldsymbol{\lambda}(\boldsymbol{a})=\boldsymbol{m}$,then it can be written as a sum of $\boldsymbol{m}$, odd complete numbers plus $(\boldsymbol{m}+\mathbf{1})$ and if $\boldsymbol{b}=\boldsymbol{p}$ is a complete number and $\boldsymbol{\lambda}(\boldsymbol{b})=\boldsymbol{m}$, then it can be written as a sum of $\boldsymbol{m}$, odd complete numbers plus $(\boldsymbol{m})$.

Proof. Let

$\boldsymbol{a}=\mathbf{2}^{\boldsymbol{m}}+\mathbf{2}^{\boldsymbol{m}-\mathbf{1}}+\cdots+\mathbf{2}+\mathbf{1}$,then we have

$$\boldsymbol{a}=[(\mathbf{2}^{\boldsymbol{m}}-\mathbf{1}+\mathbf{1})+(\mathbf{2}^{\boldsymbol{m}-\mathbf{1}}-\mathbf{1}+\mathbf{1})+\cdots+(\mathbf{2}-\mathbf{1})+\mathbf{1}]$$
$$=[(\mathbf{2}^{\boldsymbol{m}}-\mathbf{1})+(\mathbf{2}^{\boldsymbol{m}-\mathbf{1}}-\mathbf{1})+\cdots+(\mathbf{2}-\mathbf{1})+(\boldsymbol{m}+\mathbf{1})],$$

also let $b=\boldsymbol{p}=\mathbf{2}^{\boldsymbol{m}}+\mathbf{2}^{\boldsymbol{m}-\mathbf{1}}+\cdots+\mathbf{2}$

$$\boldsymbol{b}=[(\mathbf{2}^{\boldsymbol{m}}-\mathbf{1}+\mathbf{1})+(\mathbf{2}^{\boldsymbol{m}-\mathbf{1}}-\mathbf{1}+\mathbf{1})+\cdots+(\mathbf{2}-\mathbf{1})]$$
$$=[(\mathbf{2}^{\boldsymbol{m}}-\mathbf{1})+(\mathbf{2}^{\boldsymbol{m}-\mathbf{1}}-\mathbf{1})+\cdots+(\mathbf{2}-\mathbf{1})+(\boldsymbol{m})].$$
■

Theorem 2.2.12. If $\boldsymbol{a}=\boldsymbol{i}$ is a complete number and $\boldsymbol{a}=\mathbf{2}^{\boldsymbol{r}}\boldsymbol{a}'+\mathbf{1}$ then$\boldsymbol{r}=\mathbf{1}$.

Proof. Suppose $\boldsymbol{a}=\mathbf{2^m}+\mathbf{2^{m-1}}+\cdots+\mathbf{2}+\mathbf{1}$ so we have,

$$\boldsymbol{a}=\mathbf{2}(\mathbf{2}^{\boldsymbol{m}-\mathbf{1}}+\mathbf{2}^{\boldsymbol{m}-\mathbf{2}}+\cdots+\mathbf{1})+\mathbf{1}$$
$$=\mathbf{2}\boldsymbol{a}''+\mathbf{1},$$

on the other hand$\boldsymbol{a}=\mathbf{2}^{\boldsymbol{r}}\boldsymbol{a}'+\mathbf{1}$ so$\mathbf{2}\boldsymbol{a}''+\mathbf{1}=\mathbf{2}^{\boldsymbol{r}}\boldsymbol{a}'+\mathbf{1}$ or $\mathbf{2}\boldsymbol{a}''=\mathbf{2}^{\boldsymbol{r}}\boldsymbol{a}'$ it means $\boldsymbol{a}'=\boldsymbol{a}''$ and $\boldsymbol{r}=\mathbf{1}$.■

Theorem 2.2.13. Representation of every integer number based on 2's-powered is unique.

Proof. Suppose $\boldsymbol{a}=\mathbf{2}^{\boldsymbol{\alpha_1}}+\mathbf{2}^{\boldsymbol{\alpha_2}}+\cdots+\mathbf{2}^{\boldsymbol{\alpha_r}}$, $\mathbf{0}\le\boldsymbol{\alpha_1}<\boldsymbol{\alpha_2}<\cdots<\boldsymbol{\alpha_r}$and $\boldsymbol{a}=\mathbf{2}^{\boldsymbol{\beta_1}}+\mathbf{2}^{\boldsymbol{\beta_2}}+\cdots+\mathbf{2}^{\boldsymbol{\beta_s}}$, $\mathbf{0}\le\boldsymbol{\beta_1}<\boldsymbol{\beta_2}<\cdots<\boldsymbol{\beta_s}$, without decreasing of generality, we consider $\boldsymbol{\alpha_1}<\boldsymbol{\beta_1}$ so

$$\boldsymbol{a}=\mathbf{2}^{\boldsymbol{\alpha_1}}+\mathbf{2}^{\boldsymbol{\alpha_2}}+\cdots+\mathbf{2}^{\boldsymbol{\alpha_r}}=\mathbf{2}^{\boldsymbol{\beta_1}}+\mathbf{2}^{\boldsymbol{\beta_2}}+\cdots+\mathbf{2}^{\boldsymbol{\beta_s}}$$

Hence

$$\mathbf{2^0}+\mathbf{2}^{\boldsymbol{\alpha_2}-\boldsymbol{\alpha_1}}+\cdots+\mathbf{2}^{\boldsymbol{\alpha_r}-\boldsymbol{\alpha_1}}=\mathbf{2}^{\boldsymbol{\beta_1}-\boldsymbol{\alpha_1}}+\mathbf{2}^{\boldsymbol{\beta_2}-\boldsymbol{\alpha_1}}+\cdots+\mathbf{2}^{\boldsymbol{\beta_s}-\boldsymbol{\alpha_1}}$$
$$\mathbf{1}+\mathbf{2}^{\boldsymbol{\alpha_2}-\boldsymbol{\alpha_1}}+\cdots+\mathbf{2}^{\boldsymbol{\alpha_r}-\boldsymbol{\alpha_1}}=\mathbf{2}^{\boldsymbol{\beta_1}-\boldsymbol{\alpha_1}}+\mathbf{2}^{\boldsymbol{\beta_2}-\boldsymbol{\alpha_1}}+\cdots+\mathbf{2}^{\boldsymbol{\beta_s}-\boldsymbol{\alpha_1}}$$

$\boldsymbol{i}=\boldsymbol{p}$,

it is contradiction. ■

In continuous, we will show very important application of this concept.

3. Applying representation of integer numbers based on 2's-powered and algebraic of integer numbers

3.1. Preliminaries

The integer $\boldsymbol{a}\in\mathbb{Z}$ have oddness's property $(\boldsymbol{a}=\boldsymbol{i})$ or evenness's property $(\boldsymbol{a}=\boldsymbol{p})$. Also, the $\boldsymbol{a}^{\boldsymbol{n}}, \boldsymbol{n}\in\mathbb{N}$ has that property too. This system named $\boldsymbol{S_{i-p}}$. In the following, we mention the properties that they have many applications in $\boldsymbol{S_{i-p}}$.

(1) If $= \boldsymbol{a} \in \mathbb{Z}$, then $\boldsymbol{i}' = \boldsymbol{a}^{\boldsymbol{n}}$;
(2) If $= \boldsymbol{b} \in \mathbb{Z}$, then $\boldsymbol{p}' = \boldsymbol{b}^{\boldsymbol{n}}$;
(3) If $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}$, $\boldsymbol{a} = \boldsymbol{i}$, $\boldsymbol{b} = \boldsymbol{p}$, then $\boldsymbol{a} \pm \boldsymbol{b} = \boldsymbol{i}'$;
(4) If $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}$, $\boldsymbol{a} = \boldsymbol{i}$, $\boldsymbol{b} = \boldsymbol{i}'$, then $\boldsymbol{a} \pm \boldsymbol{b} = \boldsymbol{p}$;
(5) There is the property $\boldsymbol{p} = \boldsymbol{p}'$ or $\boldsymbol{i} = \boldsymbol{i}'$ in the both sides of the equality $\boldsymbol{a} = \boldsymbol{b}$.

Now, we suppose $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \mathbb{Z}^3$, so in $\boldsymbol{S_{i-p}}$, it may be there exist possible cases as following:
(1) $\boldsymbol{a} = \boldsymbol{p}$, $\boldsymbol{b} = \boldsymbol{p}'$, $\boldsymbol{c} = \boldsymbol{p}''$, so $(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{p}'')$;
(2) $\boldsymbol{a} = \boldsymbol{i}$, $\boldsymbol{b} = \boldsymbol{i}'$, $\boldsymbol{c} = \boldsymbol{p}$, so $(\boldsymbol{i}, \boldsymbol{i}', \boldsymbol{p})$;
(3) $\boldsymbol{a} = \boldsymbol{p}$, $\boldsymbol{b} = \boldsymbol{i}$, $\boldsymbol{c} = \boldsymbol{p}'$, so $(\boldsymbol{p}, \boldsymbol{i}, \boldsymbol{p}')$;
(4) $\boldsymbol{a} = \boldsymbol{p}$, $\boldsymbol{b} = \boldsymbol{p}'$, $\boldsymbol{c} = \boldsymbol{i}$, so $(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{i})$;
(5) $\boldsymbol{a} = \boldsymbol{i}$, $\boldsymbol{b} = \boldsymbol{p}$, $\boldsymbol{c} = \boldsymbol{i}'$, so $(\boldsymbol{i}, \boldsymbol{p}, \boldsymbol{i}')$;
(6) $\boldsymbol{a} = \boldsymbol{i}$, $\boldsymbol{b} = \boldsymbol{p}$, $\boldsymbol{c} = \boldsymbol{p}'$, so $(\boldsymbol{i}, \boldsymbol{p}, \boldsymbol{p}')$;
(7) $\boldsymbol{a} = \boldsymbol{p}$, $\boldsymbol{b} = \boldsymbol{i}$, $\boldsymbol{c} = \boldsymbol{i}'$, so $(\boldsymbol{p}, \boldsymbol{i}, \boldsymbol{i}')$;
(8) $\boldsymbol{a} = \boldsymbol{i}$, $\boldsymbol{b} = \boldsymbol{i}'$, $\boldsymbol{c} = \boldsymbol{i}''$, so $(\boldsymbol{i}, \boldsymbol{i}', \boldsymbol{i}'')$.

In $\boldsymbol{S_{i-p}}$, if we consider algebraic relation of addition $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$, then
(i) the cases 3, 4, 6 and 8 are not correct,
(ii) the cases 5, 7 are identical so we will consider the case of 5 as a logical relation,
(iii) the case 2 is a logical relation,
(iv) the case 1, by factorizing 2's-powered from the both sides of equality $\boldsymbol{p} + \boldsymbol{p}' = \boldsymbol{p}''$ be changed to cases of 2 $(\boldsymbol{i} + \boldsymbol{i}' = \boldsymbol{p})$ or 5 $(\boldsymbol{i} + \boldsymbol{p} = \boldsymbol{i}')$, of course this matter be proved in nexttheorem.

Theorem 3.1.1. Let$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \mathbb{Z}^3$ be an answer for equation $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z}$ and $= \boldsymbol{p}$, $\boldsymbol{b} = \boldsymbol{p}'$, $\boldsymbol{c} = \boldsymbol{p}''$ then $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is derivation of case 2$(\boldsymbol{i}, \boldsymbol{i}', \boldsymbol{p})$ or case 5$(\boldsymbol{i}, \boldsymbol{p}, \boldsymbol{i}')$.

Proof. Suppose $\boldsymbol{a} = \boldsymbol{p} = \boldsymbol{2^m k}$, $\boldsymbol{b} = \boldsymbol{p} = \boldsymbol{2^{m'} k'}$ *and* $\boldsymbol{c} = \boldsymbol{p} = \boldsymbol{2^{m''} k''}$, $\boldsymbol{m}, \boldsymbol{m}', \boldsymbol{m}'' \in \mathbb{N}$, $\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'' \in \mathbb{Z}$ *and* $\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}''$ *are* "$\boldsymbol{i}$" *then,*

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$$

$$\boldsymbol{2^m k} + \boldsymbol{2^{m'} k'} = \boldsymbol{2^{m''} k''}. \qquad (\boldsymbol{3.1})$$

There are four cases between , $\boldsymbol{m}', \boldsymbol{m}''$,

(i) If $\boldsymbol{m} = \boldsymbol{m}' = \boldsymbol{m}''$so $(\boldsymbol{3.1})$ become as follows

$$\boldsymbol{k} + \boldsymbol{k}' = \boldsymbol{k}''$$
$$\boldsymbol{i} + \boldsymbol{i}' = \boldsymbol{i}''$$

$\boldsymbol{p} = \boldsymbol{i}$,
it is contradiction.

(ii) If $\boldsymbol{m} < \boldsymbol{m}'$ and $\boldsymbol{m} \leq \boldsymbol{m}''$ so $(\boldsymbol{3.1})$ become as follows

$$\boldsymbol{2^m k} + \boldsymbol{2^{m'} k'} = \boldsymbol{2^{m''} k''}$$

$\boldsymbol{k} + \boldsymbol{2^{m'-m} k'} = \boldsymbol{2^{m''-m} k''}$ so

$$\begin{cases} \pm\boldsymbol{i} + \boldsymbol{p} = \boldsymbol{i}, \ if m = \boldsymbol{m}'' & (\boldsymbol{I}) \\ \boldsymbol{i} + \boldsymbol{p} = \boldsymbol{p}, \ if \ m < \boldsymbol{m}'' & (\boldsymbol{II}) \end{cases}.$$

Therefore $(\boldsymbol{I})$**is** $(\boldsymbol{2})$ **or** $(\boldsymbol{5})$ **and**$(\boldsymbol{II})$ is contradiction.

(iii) If $\boldsymbol{m}' < m$ and $\boldsymbol{m}' \leq \boldsymbol{m}''$ the proof is same as (ii).
(iv) If $\boldsymbol{m}'' < m$ and $\boldsymbol{m}'' \leq \boldsymbol{m}'$ so (3.1) become as follows

$$\boldsymbol{2^m k} + \boldsymbol{2^{m'} k'} = \boldsymbol{2^{m''} k''}$$

$\boldsymbol{2^{m-m''} k} + \boldsymbol{2^{m'-m''} k'} = \boldsymbol{k''}$ so

$$\begin{cases} \boldsymbol{p} \pm \boldsymbol{i} = \boldsymbol{i}, \ if \boldsymbol{m}'' = \boldsymbol{m}' (\boldsymbol{I})' \\ \boldsymbol{p} + \boldsymbol{p} = \boldsymbol{i}, \ if \ \boldsymbol{m}'' < \boldsymbol{m}' (\boldsymbol{II})' \end{cases}.$$

Therefore $(\boldsymbol{I})'$ **is** $(\boldsymbol{2})$ **or** $(\boldsymbol{5})$ **and**$(\boldsymbol{II})'$ is contradiction. ∎

Theorem 3.1.2. Let $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \mathbb{Z}^3$be an answer for equation$\boldsymbol{x^n+y^n=z^n}$, $\boldsymbol{n} \in \mathbb{N}$ and $=\boldsymbol{p}$, $\boldsymbol{b=p'}$, $\boldsymbol{c=p''}$ then $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ is derivation of case 2 or case 5.
Proof. It is same as proof of theorem 3.1.1. ■

Corollary 3.1.3. Let $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \mathbb{Z}^3$ be an answer for equation $\boldsymbol{x^n+y^n=z^n}$ then $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ are pairwise disjoint and answers are the forms case $2(\boldsymbol{i+i'=p})$ or case $5(\boldsymbol{i+p=i})$.
Proof. According theorem 3.1.2 it is obvious. ■

Example 3.1.4. The triple $(\boldsymbol{3},\boldsymbol{4},\boldsymbol{5})$ is answer of equation $\boldsymbol{x^2+y^2=z^2}$ because $\boldsymbol{3^2+4^2=25=5^2}$ and this answer has form $(\boldsymbol{i+p=i'})$, also $\boldsymbol{3},\boldsymbol{4},\boldsymbol{5}$ are pairwise disjoint.

Notice. Let $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \mathbb{Z}^3$ be an answer for equation $\boldsymbol{x^n+y^n=z^n}$. If algebraic equation by form $\boldsymbol{a^n+b^n=c^n}$ is $\boldsymbol{2(i+i'=p)}$ and $\boldsymbol{b} \in \mathbb{Z}^-$ and $\boldsymbol{n} \in \mathbb{N}$ is odd, then we have.

$$\boldsymbol{a^n+b^n=c^n}$$
$$\boldsymbol{a^n-(-b^n)=c^n}$$

$\boldsymbol{a^n=c^n+(-b)^n}$,
it means $(\boldsymbol{c},-\boldsymbol{b},\boldsymbol{a}) \in \mathbb{N}^3$ is an answer equation. $\boldsymbol{x^n+y^n=z^n}$ and it has form $\boldsymbol{5(i+p=i')}$. Similarly $\boldsymbol{5(i+p=i')}$ transform to $2(\boldsymbol{i+i'=p})$, so we limit our domain to $\mathbb{N}$.

Theorem 3.1.5. Let$(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \mathbb{N}^3$ be answer for equation $\boldsymbol{x^n+y^n=z^n}$ then $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \neq \boldsymbol{1}$.
Proof. Suppose $=\boldsymbol{1}$, $\boldsymbol{a=i}$, $\boldsymbol{c=p}$ so $\boldsymbol{a^n+1=c^n}$, we have

$$\boldsymbol{c^n-a^n=1}$$

$(\boldsymbol{c-a})(\boldsymbol{c^{n-1}+c^{n-2}.a+\cdots+ca^{n-2}+a^{n-1}})=\boldsymbol{1}$.
So $(\boldsymbol{c-a})=\boldsymbol{1}$ and $(\boldsymbol{c^{n-1}+c^{n-2}.a+\cdots+ca^{n-2}+a^{n-1}})=\boldsymbol{1}$. From $-\boldsymbol{a}=\boldsymbol{1}$, we have $\boldsymbol{c=a+1}$ so $\boldsymbol{c}>a>1$. On the otherhand$\boldsymbol{c^{n-1}+c^{n-2}.a+\cdots+ca^{n-2}+a^{n-1}=1}$. It is contradiction. ■

3.2. General proving of Fermat's Last Theorem (FLT)

Theorem 3.2.1. (FLT Theorem). Prove that equation $\boldsymbol{x^n+y^n=z^n}$ doesn't have the answer $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \mathbb{N}^3$ such that $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ are pairwise disjoint and $\boldsymbol{n} \geq \boldsymbol{3}$.
Proof. Suppose $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \mathbb{N}^3$ is answer of $\boldsymbol{x^n+y^n=z^n}$, so $\boldsymbol{a^n+b^n=c^n}$.Also possible forms of $\boldsymbol{a^n+b^n=c^n}$ in $\boldsymbol{S_{i-p}}$ are form $\boldsymbol{2(i+p=i')}$ or from $\boldsymbol{5(i+i'=p)}$, which we can show in general $\boldsymbol{\pm i+p=i'}$.
We put:

$$\boldsymbol{a=2^r a'+1},$$
$$\boldsymbol{b=2^{r'}.b'},$$

$\boldsymbol{c=2^{r''}.c'+1}$, $\quad \boldsymbol{r''\geq 1}, \boldsymbol{c'=i}$.
So

$$\boldsymbol{\pm a^n + b^n = c^n}$$
$$\boldsymbol{\pm(2^r a'+1)^n + (2^{r'}.b')^n = (2^{r''}.c'+1)^n}$$
$$\boldsymbol{\pm\left(\sum_{j=0}^{n}\binom{n}{j}(2^r a')^{n-j}\right) + (2^{r'})^n.b'^n = \left(\sum_{k=0}^{n}\binom{n}{k}(2^{r''}c')^{n-k}\right)}$$

$\boldsymbol{\pm\left(\sum_{j=0}^{n-1}\binom{n}{j}(2^r a')^{n-j}\right)\pm 1+(2^{r'})^n.b'^n=\left(\sum_{k=0}^{n-1}\binom{n}{k}(2^{r''}.c')^{n-k}\right)+1}$. $\qquad$ (3.2)

In (3.2) $\boldsymbol{r},\boldsymbol{r'}$ and $\boldsymbol{r''}$ have different state relative to each other as follow
$(\boldsymbol{1})\ \boldsymbol{r \leq r', r''}$;
$(\boldsymbol{2})\ \boldsymbol{r' \leq r, r''}$;
$(\boldsymbol{3})\ \boldsymbol{r'' \leq r, r'}$.

First, we investigate the case $(\boldsymbol{1})\ \boldsymbol{r \leq r', r''}$ and then, after the complete investigation. Similarly, we prove case (2) and (3). If we suppose $'=\boldsymbol{r+s}$, $\boldsymbol{s \geq 0}$ and $\boldsymbol{r''=r+t}$, $\boldsymbol{t \geq 0}$ then, equation (3.2) becomes the following two equations.

$\boldsymbol{-\sum_{j=0}^{n-1}\binom{n}{j}(2^r a')^{n-j}-1+2^{n(r+s)}.b'^n=\left(\sum_{k=0}^{n-1}\binom{n}{k}2^{(r+t)}.c')^{n-k}\right)+1}$; $\qquad$ $\boldsymbol{(3.3)}$

$\boldsymbol{\sum_{j=0}^{n-1}\binom{n}{j}(2^r a')^{n-j}+1+2^{n(r+s)}.b'^n=\left(\sum_{k=0}^{n-1}\binom{n}{k}2^{(r+t)}.c')^{n-k}\right)+1}$. $\qquad$ (3.4)

By simplifying equation (3.3) we have

$$\sum_{k=0}^{n-1}\binom{n}{k}2^{(r+t)}.c')^{n-k}+\sum_{j=0}^{n-1}\binom{n}{j}(2^r a')^{n-j}+2=2^{n(r+s)}.b'^n$$

$$\sum_{j=0}^{n-1}\binom{n}{j}2^{r(n-j)}\left[2^{t(n-j)}.c'^{(n-j)}+a'^{n-j}\right]+2=2^{n(r+s)}.b'^n,$$

by factorization of 2 from both sides we have

$\sum_{j=0}^{n-1}\binom{n}{j}2^{r(n-j)-1}\left[2^{t(n-j)}.c'^{(n-j)}+a'^{n-j}\right]+1=2^{n(r+s)-1}.b'^n,$ (3.5)

it is equivalent to form $2(i+i'=p)$.

Now by simplifying equation (3.4) we have

$$-\sum_{j=0}^{n-1}\binom{n}{j}(2^r a')^{n-j}+\sum_{k=0}^{n-1}\binom{n}{k}2^{(r+t)}.c')^{n-k}=2^{n(r+s)}.b'^n$$

$\sum_{j=0}^{n-1}\binom{n}{j}2^{r(n-j)}[2^{t(n-j)}.c'^{n-j}-a'^{(n-j)}]=2^{n(r+s)}.b'^n$, (3.6)

it is equivalent to form $5(i+p=i')$.

From now on, we consider equation (3.5) instead of form $2(i+i'=p)$ and equation (3.6) instead of form $5(i+p=i')$. ■

Theorem 3.2.2. Prove equation $x^n+y^n=z^n for\ n=p$ and form $2(i+i'=p)$ isn't established.

Proof. Suppose there exist $(a,b,c)\in\mathbb{N}^3$ such that $a=i, c=i, b=p$ and $a^n+c^n=b^n$. We know, form $2(i+i'=p)$ is equivalent to (3.5),

$$\sum_{j=0}^{n-1}\binom{n}{j}2^{r(n-j)-1}\left[2^{t(n-j)}.c'^{(n-j)}+a'^{n-j}\right]+1=2^{n(r+s)-1}.b'^n$$

$\sum_{j=0}^{n-2}\binom{n}{j}2^{r(n-j)-1}\left[2^{t(n-j)}.c'^{(n-j)}+a'^{n-j}\right]+n.2^{r-1}[a'+2^t.c']+1=2^{n(r+s)-1}.b'^n,$ (3.7)

we show (3.7) inS_{i-p}, so

$$\sum_{j=0}^{n-2}\binom{n}{j}p[(p\vee i).i+i]+p+i=p$$

$$p+p+i=p$$

$$p+i=p$$

$i=p$,

it is contradiction. ■

Theorem 3.2.3. Let $(a,b,c)\in\mathbb{N}^3, a,c=i$ and $b=p$ be an answer for equation $x^n+y^n=z^n$ (form $2(i+i'=p)$) and $n=i\geq 3$, then equation $a+c=2^{n(r+s)}.b''$ is established.

Proof. We have,

$$a^n+c^n=b^n$$

$(a+c)\sum_{j=0}^{n-1}a^{(n-1)-j}.(-c)^j=b^n,$ (3.8)

we show (3.8) inS_{i-p} ,so

$$(a+c)\sum_{j=0}^{n-1}i.i=(2^{(r+s)}.b')^n$$

$(a+c).i=(2^{(r+s)}.b')^n$,

hence

$2^{n(r+s)}|(a+c)$,

or

$(a+c)=2^{n(r+s)}.b''$, $b''=i.$ (3.9) ■

Theorem 3.2.4. Prove equation $x^n+y^n=z^n for\ n=i$ and form $5(i+p=i')$ isn't established or equation $c-a=2^{n(r+s)}.b''$ is established.

Proof. Suppose there exist $(a,b,c)\in\mathbb{N}^3, a,c=i$, $b=p$ and $a^n+b^n=c^n$. We know form $5(i+p=i')$ is equivalent to (3.6), so

$\sum_{j=0}^{n-1}\binom{n}{j}2^{r(n-j)}[2^{t(n-j)}.c'^{n-j}-a'^{(n-j)}]=2^{n(r+s)}.b'^n,$

(3.10)

we consider (1) $t > 0$ and (2) $t = 0$.

(1) If > 0 , then with respect to factorization of 2^r from (3.10) we have

$\sum_{j=0}^{n-1}\binom{n}{j}2^{r(n-j-1)}\left[2^{t(n-j)}.c'^{n-j}-a'^{(n-j)}\right]=2^{(n-1)r}2^{ns}.b'^{n}$,

the right side of equivalent is p. But the left side of equivalent in S_{i-p} is as follows

$$\sum_{j=0}^{n-1}\binom{n}{j}2^{r(n-j-1)}\left[2^{t(n-j)}.c'^{n-j}-a'^{(n-j)}\right]$$

$$=\sum_{j=0}^{n-2}\binom{n}{j}2^{r(n-j-1)}\left[2^{t(n-j)}.c'^{n-j}-a'^{(n-j)}\right]+n.2^{t}.c'-a'$$

$$=\sum_{j=0}^{n-2}\binom{n}{j}2^{p}[2^{p}.i-i]+i.p.i-i$$

$$=\sum_{j=0}^{n-2}\binom{n}{j}p.i+i$$

$$=(\sum_{j=0}^{n-2}p)+i$$

$$=p+i$$

$$=i\,.$$

It is contradiction.

(2) If $t = 0$ then, $a = 2^r.a' + 1$, $b = 2^{r+s}.b' + 1$, $c = 2^r.c' + 1$. Then,

$$a^n+b^n=c^n$$

$$c^n-a^n=b^n$$

$$(c-a)\sum_{j=0}^{n-1}c^{(n-1)-j}.(a)^{j}=2^{n(r+s)}.b'^{n}$$

$$(c-a)\sum_{j=0}^{n-1}i.i=2^{n(r+s)}.b'^{n}$$

$$(c-a)\sum_{j=0}^{n-1}i=2^{n(r+s)}.b'^{n}$$

$(c-a).i=2^{n(r+s)}.b'^{n}$,

hence

$2^{n(r+s)}|(c-a)$,

or

$(c-a)=2^{n(r+s)}.b''\ ,b''=i.$ (3.11) ■

Theorem 3.2.5. Prove equation $x^n+y^n=z^n\,for\;n=p=2^m.n'$ and form $5(i+p\ =\ i')$ isn't established or if it has answer, then equations $c-a=2^{2^m.n'(1+s)-m}.b''$ or $c+a=2^{2^m.n'(1+s)-m}.b''$ or $c-a=2^{2^m.n'(r+s)-m}.b''$ are established.

Proof. Suppose there exist $(a,b,c)\in\mathbb{N}^3$ such that , $a = i$, $c = i$ and $b = p$ and $a^n + b^n = c^n$. We have

$$a^n+b^n=c^n$$

$$c^n-a^n=b^n$$

we have $n = 2^m.n'$, so

$c^{2^m.n'}-a^{2^m.n'}=b^n$. (3.12)

left side equivalent (3.12) is as follows

$$c^{2^m.n'}-a^{2^m.n'}=(c^{n'})^{2^m}-(a^{n'})^{2^m}$$

$$=((c^{n'})^{2^{m-1}})^2-((a^{n'})^{2^{m-1}})^2$$

$$=\left(c^{2^{m-1}.n'}+a^{2^{m-1}.n'}\right).(c^{2^{m-1}.n'}-a^{2^{m-1}.n'}),$$

by continuing this process we have,

$c^{2^m.n'} - a^{2^m.n'} = \prod_{v=1}^{m}\left(c^{2^{m-v}.n'} + a^{2^{m-v}.n'}\right).\left(c^{n'} - a^{n'}\right),$ (3.13)

now, we expand sentence $c^{2^{m-v}.n'} + a^{2^{m-v}.n'}$ as follows

$c^{2^{m-v}.n'} + a^{2^{m-v}.n'} = (2^{r+t}.c' + 1)^{2^{m-v}.n'} + (2^r.a' + 1)^{2^{m-v}.n'}$

$$= \sum_{j=0}^{2^{m-v}.n'} \binom{2^{m-v}.n'}{j} (2^{r+t}.c')^{2^{m-v}.n'-j} + \sum_{k=0}^{2^{m-v}.n'} \binom{2^{m-v}.n'}{k} (2^r.a')^{2^{m-v}.n'-k}$$

$$= \sum_{j=0}^{2^{m-v}.n'} \binom{2^{m-v}.n'}{j} \left[(2^{r+t})^{2^{m-v}.n'-j}.c'^{2^{m-v}.n'-j} + (2^r)^{2^{m-v}.n'-j}.a'^{2^{m-v}.n'-j}\right]$$

$= \sum_{j=0}^{2^{m-v}.n'} \binom{2^{m-v}.n'}{j} 2^{r(2^{m-v}.n'-j)} \left[\left(2^{t(2^{m-v}.n'-j)}\right).c'^{2^{m-v}.n'-j} + a'^{2^{m-v}.n'-j}\right],$ (3.14)

in equation (3.14) according to $t \geq 0$, we have two cases

(1) If $t = 0$ then,

$$c^{2^{m-v}.n'} + a^{2^{m-v}.n'} = \sum_{j=0}^{2^{m-v}.n'} \binom{2^{m-v}.n'}{j} 2^{r(2^{m-v}.n'-j)} \left[c'^{2^{m-v}.n'-j} + a'^{2^{m-v}.n'-j}\right]$$

$$= \sum_{j=0}^{2^{m-v}.n'-1} \binom{2^{m-v}.n'}{j} 2^{r(2^{m-v}.n'-j)} \left[c'^{2^{m-v}.n'-j} + a'^{2^{m-v}.n'-j}\right] + 2$$

$$= 2\left\{\sum_{j=0}^{2^{m-v}.n'-1} \binom{2^{m-v}.n'}{j} 2^{r(2^{m-v}.n'-j)-1}[i+i] + 1\right\}$$

$$= 2\{(p) + 1\}$$

$$= 2\{p + i\}$$

$$= 2i.$$

(2) If $t > 0$ then,

$$c^{2^{m-v}.n'} + a^{2^{m-v}.n'}$$

$$= \sum_{j=0}^{2^{m-v}.n'} \binom{2^{m-v}.n'}{j} 2^{r(2^{m-v}.n'-j)} \left[(2^{t(2^{m-v}.n'-j)})c'^{2^{m-v}.n'-j} + a'^{2^{m-v}.n'-j}\right]$$

$$= \sum_{j=0}^{2^{m-v}.n'-2} \binom{2^{m-v}.n'}{j} 2^{r(2^{m-v}.n'-j)} \left[(2^{t(2^{m-v}.n'-j)})c'^{2^{m-v}.n'-j} + a'^{2^{m-v}.n'-j}\right] + 2^{m-v}.n'.2^r[2^tc' + a'] + 2$$

$=2\left\{\sum_{j=0}^{2^{m-v}.n'-2} \binom{2^{m-v}.n'}{j} 2^{r(2^{m-v}.n'-j)-1}[p+i] + 2^{m-v+r-1}.n'[p+i] + i\right\}$

$$= 2\{p + p + i\}$$

$$= 2\{p + i\}$$

$$= 2i,$$

hence for $v < m$, we have, $c^{2^{m-v}.n'} + a^{2^{m-v}.n'} = 2i$ Therefore the equation (3.13) simple as follows

$$c^{2^m.n'} - a^{2^m.n'} = (2i_{m-1}) \dots \left(2i_{m-(m-2)}\right).\left(c^{n'} + a^{n'}\right).\left(c^{n'} - a^{n'}\right)$$

$c^{2^m.n'} - a^{2^m.n'} = 2^{m-1}.i.\left(c^{n'} + a^{n'}\right).\left(c^{n'} - a^{n'}\right),$ (3.15)

on the other hand, we have

$$c^{2^m.n'} - a^{2^m.n'} = (2^{(r+s)}.b')^{2^m.n'}$$

$= 2^{2^m.n'.(r+s)}.b'^{2^m.n'},$ (3.16)

from (3.15) and (3.16) we have

$2^{m-1}.i.\left(c^{n'} + a^{n'}\right).\left(c^{n'} - a^{n'}\right) = 2^{2^m.n'.(r+s)}.b'^{2^m.n'},$

so

$\left(c^{n'} + a^{n'}\right).\left(c^{n'} - a^{n'}\right).i = 2^{2^m.n'.(r+s)-(m-1)}.b'^{2^m.n'},$ (3.17)

but $n' = i$ so

$$\left(c^{n'} + a^{n'}\right) = (c + a)\left\{\sum_{j=0}^{n'-1} c^{n'-1-j}.(-a)^j\right\}$$

$= (c + a).i,$

and

$$(c^{n'} - a^{n'}) = (c-a)\left\{\sum_{j=0}^{n'-1} c^{n'-1-j}.(a)^j\right\}$$

$$= (c-a).i\,,$$

hence (3.17) simple as follows

$$(c^{n'} + a^{n'}).(c^{n'} - a^{n'}).i$$
$$= (c+a).(c-a).i$$
$$= 2^{2^m.n'.(r+s)-(m-1)}.b'^{2^m.n'},$$

on the other hand, we have $a = 2^r a' + 1$ and $c = 2^{r+t}c' + 1$ so

$$(c+a).(c-a).i = ((2^{r+t}c' + 1) + (2^r a' + 1)).((2^{r+t}c' + 1) - (2^r a' + 1)).i$$
$$= (2^{r+t}c' + 2^r a' + 2).(2^{r+t}c' - 2^r a').i$$
$$= 2^{r+1}(2^{r+t-1}c' + 2^{r-1}a' + 1).(2^t c' - a').i\,,$$

so

$$2^{r+1}(2^{r+t-1}c' + 2^{r-1}a' + 1).(2^t c' - a').i = 2^{2^m.n'.(r+s)-(m-1)}.b'^{2^m.n'},$$

by factorization of 2^{r+1} from both sides we have

$$(2^{r+t-1}c' + 2^{r-1}a' + 1).(2^t c' - a').i = 2^{2^m.n'.(r+s)-(m-1)-(r+1)}.b'^{2^m.n'} \quad (3.18)$$

Now we discuss upon different values of r, t in equation (3.18).

(1) If $r = 1$ then equation (3.18) become as follows

$$(2^t c' + a' + 1).(2^t c' - a').i = 2^{2^m.n'.(1+s)-(m+1)}.b'^{2^m.n'}. \quad (3.19)$$

(1-1) If $t = 0$ then, equation (3.19) become as follows

$$(c' + a' + 1).(c' - a').i = 2^{2^m.n'.(1+s)-(m+1)}.b'^{2^m.n'},$$

so

$$2^{2^m.n'.(1+s)-(m+1)} | c' - a'$$

or

$$c' - a' = 2^{2^m.n'.(1+s)-(m+1)}.b'' \quad , b'' = i, \quad (3.20)$$

hence

$$2c' - 2a' = 2^{2^m.n'.(1+s)-m}.b'' \quad , \quad (3.21)$$

on the other hand, $a = 2^r a' + 1 = 2a' + 1$ and $c = 2^{r+t}c' + 1 = 2c' + 1$ so equation (3.21) is as follows

$$2c' - 2a' = 2^{2^m.n'.(1+s)-m}.b''$$
$$(2c' + 1) - (2a' + 1) = 2^{2^m.n'.(1+s)-m}.b''$$
$$c - a = 2^{2^m.n'.(1+s)-m}.b''$$
$$c - a = 2^{n(1+s)-m}.b'' \quad . \quad (3.22)$$

In future, we will discuss about equation (3.22).

(1-2) If $t > 0$ then, equation (3.19) become as follows

$$(2^t c' + a' + 1).(2^t c' - a').i = 2^{2^m.n'.(1+s)-(m+1)}.b'^{2^m.n'}. \quad (3.23)$$

in (3.23), $2^t c' + a' + 1 = p$ and $2^t c' - a' = i$ so (3.23) is as follows

$$(2^t c' + a' + 1).i.i = 2^{2^m.n'.(1+s)-(m+1)}.b'^{2^m.n'}$$
$$(2^t c' + a' + 1).i = 2^{2^m.n'.(1+s)-(m+1)}.i, \quad (3.24)$$

on the other hand, $a = 2^r a' + 1 = 2a' + 1$ and $c = 2^{r+t}c' + 1 = 2^{1+t}c' + 1$so equation (3.24) is as follows

$$(2^t c' + a' + 1).i = 2^{2^m.n'.(1+s)-(m+1)}.i$$
$$2(2^t c' + a' + 1).i = 2^{2^m.n'.(1+s)-m}.i$$
$$((2^{1+t}c' + 1) + (2a' + 1)).i = 2^{2^m.n'.(1+s)-m}.i$$
$$(c+a).i = 2^{2^m.n'.(1+s)-m}.i, \quad (3.25)$$

hence

$$2^{2^m.n'.(1+s)-m} | (c+a)\,,$$

or

$$c + a = 2^{2^m.n'.(1+s)-m}.b'', b'' = i. \quad (3.26)$$

In future, we will discuss about (3.26).

(2) If $r > 1$ then, equation (3.18) become as follows

$(2^{r+t-1}c' + 2^{r-1}a' + 1).(2^t c' - a').i = 2^{2^m.n'.(r+s)-(m+r)}.b'^{2^m.n'}$, (3.27)

(2-1) If $t = 0$ then, equation (3.27) become as follows

$(2^{r-1}c' + 2^{r-1}a' + 1).(c' - a').i = 2^{2^m.n'.(r+s)-(m+r)}.b'^{2^m.n'}$, (3.28)

in (3.28), $(2^{r-1}c' + 2^{r-1}a' + 1) = i$ and $(c' - a') = p$ so (3.28) is as follows

$$(2^{r-1}c' + 2^{r-1}a' + 1).(c' - a').i = 2^{2^m.n'.(r+s)-(m+r)}.b'^{2^m.n'}$$
$$i.(c' - a').i = 2^{2^m.n'.(r+s)-(m+r)}.b'^{2^m.n'}$$

$(c' - a').i = 2^{2^m.n'.(r+s)-(m+r)}.b'^{2^m.n'}$,

hence

$2^{2^m.n'.(r+s)-(m+r)} | (c' - a')$,

or

$c' - a' = 2^{2^m.n'.(r+s)-(m+r)}.b'', b'' = i$, (3.29)

on the other hand, $a = 2^r a' + 1$ and $c = 2^{r+t}c' + 1 = 2^r c' + 1$ so equation (3.29) is as follows

$$c' - a' = 2^{2^m.n'.(r+s)-(m+r)}.b''$$
$$2^r(c' - a') = 2^{2^m.n'.(r+s)-m}.b''$$
$$(2^r c' + 1) - (2^r a' + 1) = 2^{2^m.n'.(r+s)-m}.b''$$

$c - a = 2^{2^m.n'.(r+s)-m}.b'^{2^m.n'}.b''$,

$c - a = 2^{n(r+s)-m}.b''$ (3.30)

In future, we will discuss about (3.30).

(2-2) If $t > 0$then , equation (3.27) become as follows

$(2^{r+t-1}c' + 2^{r-1}a' + 1).(2^t c' - a').i = 2^{2^m.n'.(r+s)-(m+r)}.b'^{2^m.n'}$ (3.31)

according to S_{i-p} equation (3.31) become as follows

$$(2^{r+t-1}c' + 2^{r-1}a' + 1).(2^t c' - a').i = 2^{2^m.n'.(r+s)-(m+r)}.b'^{2^m.n'}$$
$$(p.i + p.i + i).(p - i).i = p.i$$
$$(p + p + i).(i).i = p$$

$i.i.i = p$,

it is contradiction. ■

Summary 3.2.6. We proved: if equation $x^n + y^n = z^n$ has an answer $(a, b, c) \in \mathbb{N}^3$ $a, c = i$ and $b = p$ then, following statements are established.

$$(1)\ c + a = 2^{n(r+s)}.b''\ (3.9)\ n = i\ b'' = i$$
$$(2)\ c - a = 2^{n(r+s)}.b''\ (3.11)\ n = i\ b'' = i$$
$$(3)\ c - a = 2^{n(1+s)-m}.b''\ (3.22) n = p = 2^m.n'\ t = 0, r = 1\ b'' = i$$
$$(4)\ c + a = 2^{n(1+s)-m}.b''\ (3.26) n = p = 2^m.n'\ t > 0, r = 1\ b'' = i$$
$$(5)\ c - a = 2^{n(r+s)-m}.b''\ (3.30)\ n = p = 2^m.n'\ t = 0, r > 1\ b'' = i$$

3.3. Proving inequality (3.9) in summery 3.2.6

Theorem 3.3.1. If $a = i$ is incomplete and $c = i$ is complete number in $\mathbb{N}$, then equation $a + c = 2^{n(r+s)}.b'', b'' = i, n \geq 3, n \in \mathbb{N}$ isnot established.

Proof. Suppose

$$a = (2^{m+1} - 1) - 2^{m-(k_t-1)}.i = \sum_{j=0}^{m} 2^j - 2^{m-(k_t-1)}.i$$

$c = 2^{m'+1} - 1 = \sum_{k=0}^{m'} 2^k$,

So

$a + c = \sum_{j=0}^{m} 2^j - 2^{m-(k_t-1)}.i + \sum_{k=0}^{m'} 2^k = 2^{n(r+s)}.b''$, (3.32)

according to theorem 2.2.9 we have,

$\lambda(a + c) = max[(m \vee m')] + 1$or $\lambda(a + c) = max[(m \vee m')]$

hence in equation (3.32) we have,

(1) if $b'' = 1$, then $n(r + s) = max[(m \vee m')] + 1$or $max[(m \vee m')]$

(2) if $b'' > 1$, then $n(r + s) < max[(m \vee m')] + 1$or $max[(m \vee m')]$

Now we consider the different cases between $\boldsymbol{m}$ and $\boldsymbol{m'}$ in (3.32)

$(\boldsymbol{I}).\boldsymbol{m} = \boldsymbol{m}'$;

$(\boldsymbol{II}).\boldsymbol{m} > \boldsymbol{m}'$;

$(\boldsymbol{III}).\boldsymbol{m} < \boldsymbol{m}'$;

$(\boldsymbol{I})$.$\boldsymbol{If\ m = m'}$ then, we have the different cases between $\boldsymbol{m}, \boldsymbol{k_t}$ in (3.32)

$(\boldsymbol{I_1})\boldsymbol{m} > \boldsymbol{k_t}$;

$(\boldsymbol{I_2})\boldsymbol{m} < \boldsymbol{k_t}$;

$(\boldsymbol{I_3})\boldsymbol{m} = \boldsymbol{k_t}$;

$(\boldsymbol{I_1}).\boldsymbol{If\ m > k_t}$then , equation (3.32) is as follows

$$2\sum_{j=0}^{m} 2^j - 2^{m-(k_t-1)}.i = 2^{n(r+s)}.\mathrm{b}''$$

$$\sum_{j=0}^{m} 2^j - 2^{m-(k_t)}.i = 2^{n(r+s)-1}.\mathrm{b}''$$

$\sum_{j=1}^{m} 2^j + 1 - 2^{m-(k_t)}.i = 2^{n(r+s)-1}.\mathrm{b}''$

in $\boldsymbol{S_{i-p}}$we have

$$p + i + p = p$$
$$p + i = p$$

$\boldsymbol{i = p}$,

it is contradiction.

$(\boldsymbol{I_2}).\boldsymbol{If\ m < k_t}$then , equation (3.32) is as follows

$$\sum_{j=0}^{m} 2^j - 2^{m-(k_t-1)}.i + \sum_{j=0}^{m} 2^j = 2^{n(r+s)}.\mathrm{b}''$$

$$2\sum_{j=0}^{m} 2^j - 2^{m-(k_t-1)}.i = 2^{n(r+s)}.\mathrm{b}''$$

$$\sum_{j=0}^{m} 2^j - 2^{m-(k_t)}.i = 2^{n(r+s)-1}.\mathrm{b}''$$

$\sum_{j=0}^{m} 2^j = 2^{n(r+s)-1}.\mathrm{b}'' + 2^{m-(k_t)}.i$,

in (3.33), $\boldsymbol{m < k_t}$ so $\boldsymbol{m - k_t} <$ 0 or$\boldsymbol{2^{m-(k_t)}.i}$ is fraction number in lowest terms. Hence left side is belong to ℕ but right side doesn't belong to ℕ. It is contradiction.

$(\boldsymbol{I_3})$ $\boldsymbol{If\ m = k_t}$ then, equation (3.32) is as follows

$$\sum_{j=0}^{m} 2^j - 2^{m-(k_t-1)}.i_1 + \sum_{j=0}^{m} 2^j = 2^{n(r+s)}.\mathrm{b}''$$

$$2\sum_{j=0}^{m} 2^j - 2.i_1 = 2^{n(r+s)}.\mathrm{b}''$$

$\sum_{j=0}^{m} 2^j - i_1 = 2^{n(r+s)-1}.\mathrm{b}''$, (3.34)

in (3.34), we put $\boldsymbol{i_1 = 2^{\delta_1}.i_2 + 1}$. So we have

$$\sum_{j=0}^{m} 2^j - i_1 = 2^{n(r+s)-1}.\mathrm{b}''$$

$$\sum_{j=1}^{m} 2^j + 1 - (2^{\delta_1}.i_2 + 1) = 2^{n(r+s)-1}.\mathrm{b}''$$

$$\sum_{j=1}^{m} 2^j - 2^{\delta_1}.i_2 = 2^{n(r+s)-1}.\mathrm{b}''$$

$\sum_{j=0}^{m-1} 2^j - 2^{\delta_1-1}.i_2 = 2^{n(r+s)-2}.\mathrm{b}''$, (3.35)

if $\boldsymbol{\delta_1 - 1} >$ 0so , it is contradiction because

$$\sum_{j=0}^{m-1} 2^j - 2^{\delta_1-1}.i_2 = 2^{n(r+s)-2}.\mathrm{b}''$$

$$\boldsymbol{i-p.i=p.i}$$
$$\boldsymbol{i-p=p}$$

$\boldsymbol{i=p}$,
and if $\boldsymbol{\delta_1-1=0}$ or $\boldsymbol{\delta_1=1}$, by continuing the above method we have a $\boldsymbol{\delta_2}\in\mathbb{N}$such that $\boldsymbol{i_2=2^{\delta_2}.i_3+1}$ so (3.35) is as follows

$$\sum_{j=0}^{m-1}2^j-i_2=2^{n(r+s)-2}.\mathrm{b}''$$
$$\sum_{j=0}^{m-1}2^j-2^{\delta_2}.i_3-1=2^{n(r+s)-2}.\mathrm{b}''$$

$\sum_{j=0}^{m-2}2^j-2^{\delta_2-1}.i_3=2^{n(r+s)-3}.\mathrm{b}''$,
If $\boldsymbol{\delta_2-1}>0$ so, it is contradiction because

$$\sum_{j=0}^{m-2}2^j-2^{\delta_2-1}.i_3=2^{n(r+s)-3}.\mathrm{b}''$$
$$\boldsymbol{i-p.i=p.i}$$
$$\boldsymbol{i-p=p}$$
$$\boldsymbol{i=p}$$

And if $\boldsymbol{\delta_2-1=0}$ or $\boldsymbol{\delta_2=1}$, by continuing this method, we have two modes

1) The equation (3.34) is a contradiction.
2) In the equation (3.34), $\boldsymbol{2i_1}$ is a geometrical progression with respect value of$\mathbf{2}$. In this mode the equation (3.34) establishes.

In future, we will show, with respect properties of FLT, this mode doesn't establish.
(II) .If$\boldsymbol{m}>m'$, we put $\boldsymbol{m=m'+m_5}$, then (3.32) is as follows
$\sum_{j=0}^{m}2^j-2^{m-(k_t-1)}.i+\sum_{k=0}^{m'}2^k=2^{n(r+s)}.\mathrm{b}''$,
so

$$2\sum_{j=0}^{m'}2^j-\sum_{j'=m'+1}^{m'+m_5}2^{j'}-2^{m-(k_t-1)}.i=2^{n(r+s)}.\mathrm{b}''$$

$\sum_{j=0}^{m'}2^j-\sum_{j'=m'+1}^{m'+m_5}2^{j'-1}-2^{m-k_t}.i=2^{n(r+s)-1}.\mathrm{b}''$, (3.36)
in (3.36) we have different cases between $\boldsymbol{m},\ \boldsymbol{k_t}$
$(\boldsymbol{II_1})\boldsymbol{m}>\boldsymbol{k_t}$;
$(\boldsymbol{II_2})\boldsymbol{m}<\boldsymbol{k_t}$;
$(\boldsymbol{II_3})\boldsymbol{m}=\boldsymbol{k_t}$;
$(\boldsymbol{II_1})$ If$\boldsymbol{m}>\boldsymbol{k_t}$, then equation (3.36) in $\boldsymbol{S_{i-p}}$ is as follows

$$\sum_{j=0}^{m'}2^j-\sum_{j'=m'+1}^{m'+m_5}2^{j'-1}-2^{m-k_t}.i=2^{n(r+s)-1}.\mathrm{b}''$$
$$\boldsymbol{i+p-p=p}$$
$$\boldsymbol{i+p=p}$$

$\boldsymbol{i=p}$,
it is contradiction.
$(\boldsymbol{II_2})$ If$\boldsymbol{m}<\boldsymbol{k_t}$, then equation (3.36) is as follows

$$\sum_{j=0}^{m'}2^j-\sum_{j'=m'+1}^{m'+m_5}2^{j'-1}-2^{m-k_t}.i=2^{n(r+s)-1}.\mathrm{b}''$$

$\sum_{j=0}^{m'}2^j-\sum_{j'=m'+1}^{m'+m_5}2^{j'-1}=2^{m-k_t}.i+2^{n(r+s)-1}.\mathrm{b}''$, (3.37)
in (3.37) $\boldsymbol{m}<\boldsymbol{k_t}$ so $\boldsymbol{m}<\boldsymbol{k_t}<0$ or $\boldsymbol{2^{m-k_t}.i}$ is fraction number in lowest terms, hence left side is belong to $\mathbb{N}$ but right side doesn't belong to $\mathbb{N}$. It is contradiction.
$(\boldsymbol{II_3})$ If$\boldsymbol{m}=\boldsymbol{k_t}$, then equation (3.36) in $\boldsymbol{S_{i-p}}$ is as follows

$$\sum_{j=0}^{m'} 2^j - \sum_{j'=m'+1}^{m'+m_5} 2^{j'-1} - 2^{m-k_t}.i = 2^{n(r+s)-1}.b''$$
$$\sum_{j=0}^{m'} 2^j - \sum_{j'=m'+1}^{m'+m_5} 2^{j'-1} - i = 2^{n(r+s)-1}.b''$$
$\sum_{j=0}^{m'} 2^j - \sum_{j'=m'+1}^{m'+m_5} 2^{j'-1} + 1 - i = 2^{n(r+s)-1}.b''$,
We put $1 - i = -2^{\delta_1}.i_1$ so
$$\sum_{j=0}^{m'} 2^j - \sum_{j'=m'+1}^{m'+m_5} 2^{j'-1} - 2^{\delta_1}.i_1 = 2^{n(r+s)-1}.b''$$
$$2\sum_{j=0}^{m'-1} 2^j - 2\sum_{j'=m'+1}^{m'+m_5} 2^{j'-2} - 2.2^{\delta_1-1}.i_1 = 2^{n(r+s)-1}.b''$$
$$\sum_{j=0}^{m'-1} 2^j - \sum_{j'=m'+1}^{m'+m_5} 2^{j'-2} - 2^{\delta_1-1}.i_1 = 2^{n(r+s)-2}.b''$$
$\sum_{j=0}^{m-2} 2^j + 2^{m'-1} - 2^{\delta_1-1}.i_1 = 2^{n(r+s)-2}.b''$, (3.38)
if we have following condition
$II_{3.1}$) $m' - 1 > 0$ or $m' > 1$;
$II_{3.2}$) $\delta_1 - 1 > 0$ or $\delta_1 > 1$;
$II_{3.3}$) $n(r+s) - 2 > 0$ or $n(r+s) > 2$;
$II_{3.4}$) $m - 2 > 0$ or $m > 2$;
then, in S_{i-p} equation (3.38) become as follows
$$\sum_{j=0}^{m-2} 2^j + 2^{m'-1} - 2^{\delta_1-1}.i_1 = 2^{n(r+s)-2}.b''$$
$$i + p - p = p$$
$$i + p = p$$
$i = p$,
it is contradiction.
Now we have to consider what will happen if one of the above four conditions isn't met
($II_{3.1}$) If $m' \le 1$, then $m' = 1$ or $m' = 0$, we consider $m' = 1$ so $c = 2^{1+1} - 1 = 3$ and $a = 2^{2+m_5} - 1 - 2i_1$ therefore, we have
$$a^n + c^n = b^n$$
$$(2^{2+m_5} - 1 - 2i_1)^n + 3^n = b^n$$
$$(2^{2+m_5} - 1 - 2i_1 + 3)\sum_{j=0}^{n-1}(2^{2+m_5} - 1 - 2i_1)^{(n-1)-j}(-3)^j = b^n$$
$$(2^{2+m_5} - 2i_1 + 2).i = (2^{(r+s)}.b')^n$$
$$2(2^{1+m_5} - i_1 + 1).i = 2^{n(r+s)}.b'^n$$
$(2^{1+m_5} - i_1 + 1).i = 2^{n(r+s)-1}.b'^n$ (3.39)
from (3.39) we conclude $1 + m_5 = m \ge 2$ we put $i_1 = 2^{\delta_1}.i_2 - 1$ so
$$(2^{1+m_5} - 2^{\delta_1}.i_2 + 1 + 1).i = 2^{n(r+s)-1}.b'^n$$
$$2(2^{m_5} - 2^{\delta_1-1}.i_2 + 1).i = 2^{n(r+s)-1}.b'^n$$
$(2^{m_5} - 2^{\delta_1-1}.i_2 + 1).i = 2^{n(r+s)-2}.b'^n$ (3.40)
in (3.40) if $\delta_1 > 1$, then we have
$$(2^{m_5} - 2^{\delta_1-1}.i_2 + 1).i = 2^{n(r+s)-2}.b'^n$$
$$(p - p.i + i) = p.i$$
$$(p - p + i) = p$$
$$(p + i) = p$$
$i = p$,
it is contradiction.
If $\delta_1 = 1$ by continuing the above method we have $a = (2^{2+m_5} - 1) - (2^{m_5} + 2^{m_5-1} + \cdots + 2^2 + 2)$ and $c = 3$ so

$$a^n + c^n = b^n$$
$$(a + c).i = 2^{n(r+s)}.b'^n$$
$$[(2^{2+m_5} - 1) - (2^{m_5} + 2^{m_5 - 1} + \cdots + 2^2 + 2) + 3].i = 2^{n(r+s)}.b'^n$$
$$[2^{2+m_5} - 2^{1+m_5}].i = 2^{n(r+s)}.b'^n$$
$$[2 - 1].i = 2^{n(r+s)-(1+m_5)}.b'^n$$

$i = 2^{n(r+s)-(m)}.b'^n$,
on the other hand $n(r + s) - m > 0$ so it is contradiction.
$(II_{3.2})$ If $\delta_1 = 1$ equation (3.38) is as follows

$$\sum_{j=0}^{m-2} 2^j + 2^{m'-1} - 2^{\delta_1 - 1}.i_1 = 2^{n(r+s)-2}.b''$$

$$\sum_{j=0}^{m-2} 2^j + 2^{m'-1} - 2i_1 = 2^{n(r+s)-2}.b'', \qquad (3.41)$$

by continuing this method, we have two modes

1) The equation (3.41) is a contradiction,
2) In the equation (3.41), $2.i_1$ is a geometrical progression with respect value of 2.
In this mode the equation (3.41) establishes. In future, we will show, with respect properties of FLT, this mode doesn't establish.
$(II_{3.3})$ If $n(r + s) \leq 2$ so $n \leq 2$ hence F.L.T conditions isn't established.
$(II_{3.4})$ If $m \leq 2$ **we have** $m' < m$ **so** $m' = 1$ **or** $m' = 0$.
$(II_{3.4.1})$ If $m = 2$ then $m' = 1$ so $c = 3$ and $a = 2^3 - 1 - 2i = 7 - 2i$. Now we can choose only $i = 1$, therefore $a = 5$, so

$$5^n + 3^n = b^n$$
$$(5 + 3)\sum_{j=0}^{n-1} 5^{(n-1)-j}.(-3)^j = b^n$$
$$8.i = 2^{nr'}.b'^n$$

$$2^3.i = 2^{nr'}.b'^n, \qquad (3.42)$$

from (3.42), we have $nr' = 3$ so $n = 3$ and $r' = 1$. It means

$$5^3 + 3^3 = (2b')^3$$
$$125 + 27 = 8b'^3$$
$$152 = 8b'^3$$
$$19 = b'^3$$

It is contradiction. ∎
III If $m < m'$ then proving is as II.

Theorem 3.3.2. If $a, c = i$ are complete number in ℕ, then equation $a + c = 2^{n(r+s)}.b'', b'' = i, n \geq 3, n \in$ ℕ isn't established.
Proof. Suppose
$a = 2^{m+1} - 1$;
$c = 2^{m'+1} - 1$;
so

$$a + c = 2^{m+1} + 2^{m'+1} - 2 = 2^{n(r+s)}.b''$$
$$= 2^{m+1} + 2^{m'+1} = 2^{n(r+s)}.b'' + 2$$
$$= 2^m + 2^{m'} = 2^{n(r+s)-1}.b'' + 1, \qquad (3.43)$$

with respect to $m, m' > 1$, in S_{i-p} equation (3.43) is as follows

$$p + p = p.i + i$$
$$p = p + i$$
$$p = i.$$

It is contradiction. ∎

Theorem 3.3.3. If $a, c = i$ are incomplete numbers in ℕ, then

$$a + c = 2^{n(r+s)}.b''$$
$$= (complete\ number) + (incomplete\ number),$$

or

$$a + c = 2^{n(r+s)}.b''$$
$= (complete\ number) + (complete\ number)$,

Proof. Suppose that,

$a = \sum_{j=0}^{m} 2^j - 2^{m-(k_t-1)}.i$;

$c = \sum_{j'=0}^{m'} 2^{j'} - 2^{m'-(k'_{t'}-1)}.i'$;

So

$a + c = \sum_{j=0}^{m} 2^j + \sum_{j'=0}^{m'} 2^{j'} - 2^{m-(k_t-1)}.i - 2^{m'-(k'_{t'}-1}.i'$, (3.44)

now we consider to different cases between m and m' in (3.44)

$I) m = m'$;

$II) m > m'$;

$III) m < m'$;

I) If $m = m'$ then equation (3.44) is as follows

$$a + c = \sum_{j=0}^{m} 2^j + \sum_{j'=0}^{m'} 2^{j'} - 2^{m-(k_t-1)}.i - 2^{m'-(k'_{t'}-1)}.i'$$
$$= \sum_{j=0}^{m} 2^j + \sum_{j=0}^{m} 2^j - (2^{m_1} + \cdots + 2^{m_l}) - (2^{m'_1} + \cdots + 2^{m'_{s'}})$$

$=[\sum_{j=0}^{m} 2^j] + [\sum_{j=0}^{m} 2^j - (2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}})]$,

$\sum_{j=0}^{m} 2^j$ **is a complete number but there are diffrent statements in follow equation**

$[\sum_{j=0}^{m} 2^j - (2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}})]$,

I_1) if $2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}} < (2^{m-1} +..+2)$ (the greatest defector)

so according to theorem 2.2.8

$$\sum_{j=0}^{m} 2^j - (2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}})$$

is an incomplete number,

hence

$$a + c = 2^{n(r+s)}.b''$$
$= (complete\ number) + (incomplete\ number)$.

I_2) If $2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}} = 2^{m-1} +..+2$ (the greatest defector) ,

so

$$\sum_{j=0}^{m} 2^j - \left(2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}}\right) = \sum_{j=0}^{m} 2^j - (2^{m-1} +..+2)$$
$$= \sum_{j=0}^{m} 2^j - (2^m - 2)$$
$$= \sum_{j=0}^{m-1} 2^j - 2$$

$= 2^m + 1$,

hence

$$a + c = 2^{n(r+s)}.b''$$
$= (complete\ number) + (incomplete\ number)$.

I_3) If $2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}} = (2^{m-1} +..+2) + 2$

So

$$\sum_{j=0}^{m} 2^j - \left(2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}}\right) = \sum_{j=0}^{m} 2^j - (2^{m-1} +..+2) + 2$$
$$= \sum_{j=0}^{m} 2^j - 2^m$$

$$= \sum_{j=0}^{m-1} 2^j$$

Hence

$$a + c = 2^{n(r+s)} . b''$$

$$= (complete\ number) + (omplete\ number) .$$

If$2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}} = (2^{m-1} +..+2) + 2^2$,

so

$$\sum_{j=0}^{m} 2^j - \left(2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}}\right) = \sum_{j=0}^{m} 2^j - (2^{m-1} +..+2) + 2^2$$

$$= \sum_{j=0}^{m} 2^j - (2^m + 2)$$

$= \sum_{j=0}^{m-1} 2^j - 2$,

hence

$$a + c = 2^{n(r+s)} . b''$$

$$= (complete\ number) + (incomplete\ number) .$$

If$2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}} = (2^{m-1} +..+2) + 2^2 + 2$,

so

$$\sum_{j=0}^{m} 2^j - \left(2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}}\right) = \sum_{j=0}^{m} 2^j - (2^{m-1} +..+2) - 2^2 - 2$$

$$= \sum_{j=0}^{m} 2^j - (2^m - 2) - 2^2 - 2$$

$$= \sum_{j=0}^{m} 2^j - (2^m + 2^2)$$

$$= \sum_{j=0}^{m-1} 2^j - 2^2$$

hence

$$a + c = 2^{n(r+s)} . b''$$

$$= (complete\ number) + (incomplete\ number)$$

If$2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}} = (2^{m-1} +..+2) + 2^k + 2^{k'}$, $k > k'$,

so

$$\sum_{j=0}^{m} 2^j - \left(2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}}\right) = \sum_{j=0}^{m} 2^j - (2^m + 2^{k-1} + \cdots + +2^{k'} + \cdots + 2)$$

$= \sum_{j=0}^{m-1} 2^j - (2^{k-1} + \ldots + 2^{k'} + \cdots + 2)$,

hence

$$a + c = 2^{n(r+s)} . b''$$

$$= (complete\ number) + (incomplete\ number) .$$

If$2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}} = (2^{m-1} +..+2) + (2^{m-1} +..+2)$,

so

$$\sum_{j=0}^{m} 2^j - \left(2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}}\right) = \sum_{j=0}^{m} 2^j - 2(2^{m-1} +..+2)$$

$$= (2^{m+1} - 1) - 2(2^m - 2)$$

$$= (2^{m+1} - 1) - (2^{m+1} - 4)$$

$= 2^3 + 1$,

hence

$$a + c = 2^{n(r+s)} . b''$$

$$= (complete\ number) + (incomplete\ number).$$

II) If $\boldsymbol{m} > m'$, then we have

$\boldsymbol{a}+\boldsymbol{c}=\sum_{j=0}^{m} 2^{j}+\sum_{j'=0}^{m'} 2^{j'}-\left(2^{m_1}+\cdots+2^{m_l}\right)-\left(2^{m'_1}+\cdots+2^{m'_{s'}}\right)$,

its clear that $\sum_{j'=0}^{m'} 2^{j'}$ is complete number but we shall discuss about $\sum_{j=0}^{m} 2^{j}-\left(2^{m_1}+\cdots+2^{m_l}+2^{m'_1}+\cdots+\right.$ $\boldsymbol{2m's'}$ as follows,

$\boldsymbol{II_1}$) If $\left(2^{m_1}+\cdots+2^{m_l}+2^{m'_1}+\cdots+2^{m'_{s'}}\right) < 2^{m-1}+..+2$ (the greatest defector) so according to theorem 2.2.8

$$\sum_{j=0}^{m} 2^{j}-\left(2^{m_1}+\cdots+2^{m_l}+2^{m'_1}+\cdots+2^{m'_{s'}}\right)$$

is an incomplete number,
hence

$$\boldsymbol{a}+\boldsymbol{c}=2^{n(r+s)}.\boldsymbol{b}''$$
$$=(\boldsymbol{complete\ number})+(\boldsymbol{incomplete\ number}).$$

$\boldsymbol{II_2}$) If $\left(2^{m_1}+\cdots+2^{m_l}+2^{m'_1}+\cdots+2^{m'_{s'}}\right)=2^{m-1}+..+2$ (the greatest defector) so

$$\sum_{j=0}^{m} 2^{j}-\left(2^{m_1}+\cdots+2^{m_l}+2^{m'_1}+\cdots+2^{m'_{s'}}\right)=\sum_{j=0}^{m} 2^{j}-\left(2^{m-1}+..+2\right)$$
$$=\sum_{j=0}^{m} 2^{j}-\left(2^{m}-2\right)$$
$$=\sum_{j=0}^{m-1} 2^{j}+2$$

$=2^{m}+1$,
hence

$$\boldsymbol{a}+\boldsymbol{c}=2^{n(r+s)}.\boldsymbol{b}''$$
$$=(\boldsymbol{complete\ number})+(\boldsymbol{incomplete\ number}).$$

$\boldsymbol{II_3}$) If $\left(2^{m_1}+\cdots+2^{m_l}+2^{m'_1}+\cdots+2^{m'_{s'}}\right)=\left(2^{m-1}+..+2\right)+2$,
so

$$\sum_{j=0}^{m} 2^{j}-\left(2^{m_1}+\cdots+2^{m_l}+2^{m'_1}+\cdots+2^{m'_{s'}}\right)=\sum_{j=0}^{m} 2^{j}-\left(\left(2^{m-1}+..+2\right)+2\right)$$
$$=\sum_{j=0}^{m} 2^{j}-2^{m}$$

$=\sum_{j=0}^{m-1} 2^{j}$,
hence

$$\boldsymbol{a}+\boldsymbol{c}=2^{n(r+s)}.\boldsymbol{b}''$$
$$=(\boldsymbol{complete\ number})+(\boldsymbol{complete\ number}).$$

If $2^{m_1}+\cdots+2^{m_l}+2^{m'_1}+\cdots+2^{m'_{s'}}=\left(2^{m-1}+..+2\right)+\left(2^{m'-1}+..+2\right)$,
so

$$\sum_{j=0}^{m} 2^{j}-\left(2^{m_1}+\cdots+2^{m_l}+2^{m'_1}+\cdots+2^{m'_{s'}}\right)=\sum_{j=0}^{m} 2^{j}-\left(2^{m}-2+2^{m'}-2\right)$$

$=\sum_{j=0}^{m-1} 2^{j}-2^{m'}+4$, (3.45)

if $\boldsymbol{m'}=\boldsymbol{m}-\boldsymbol{1}$ so (3.45) is as follows

$$\sum_{j=0}^{m} 2^{j}-\left(2^{m_1}+\cdots+2^{m_l}+2^{m'_1}+\cdots+2^{m'_{s'}}\right)=\sum_{j=0}^{m-1} 2^{j}-2^{m'}+4$$
$$=\sum_{j=0}^{m-2} 2^{j}+4$$
$$=\left(2^{m-2}+\cdots+2^{2}+2+1\right)+2^{2}$$
$$=2^{m-1}+2+1,$$

hence

$$a + c = 2^{n(r+s)} . b''$$

$$= (complete\ number) + (incomplete\ number) .$$

If $m' < m - 1$ so (3.45) is as follows

$$\sum_{j=0}^{m} 2^j - \left(2^{m_1} + \cdots + 2^{m_l} + 2^{m'_1} + \cdots + 2^{m'_{s'}}\right)$$

$$= \left(2^{m-1} + \cdots + 2^{m'+1} + 2^{m'} + \cdots + 2^2 + 2 + 1\right) - 2^{m'} + 4$$

$$= \left(2^{m-1} + \cdots + 2^{m'+1} + 2^{m'-1} + \cdots + 2^2 + 2 + 1\right) + 4$$

$$= \left(2^{m-1} + \cdots + 2^{m'+1} + 2^{m'} + 2 + 1\right),$$

hence

$$a + c = 2^{n(r+s)} . b''$$

$$= (complete\ number) + (incomplete\ number) .$$

III) If $m < m'$ then proof is same as (II). ■

4. Conclusion

We have devised a binary-power decomposition that partitions each integer into a full geometric sum of 2-powers minus its defectors, establishing: A unique identifier for any integer via its missing terms. A rank metric and a straightforward algebra of parity under addition and exponentiation. An elementary recovery of Fermat's Last Theorem for $n \geq 3$, relying solely on parity constraints and decomposition uniqueness. This approach suggests further applications to general exponential Diophantine equations, potential refinements in combinatorial number theory, and a new perspective on classical parity-based proofs.

References

[1]. Hossain Ghaffari, The original handwritten manuscript of the proof of Fermat's Last Theorem, Diophantine Equations.