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The Linear Transformation Matrix associated with the ZJ Transformation – Applications

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Abstract: This article presents an alternative methodology based on Functional Analysis in Finite-Dimensional Spaces, which completely dispenses with classical analytical integration. By constructing a local linear space and using the Commutation Theorem for Linear Operators, we demonstrate that it is possible to encapsulate the dynamics of the transformation kernel $e^{-\frac{z}{\beta}t}$ and the objective function within an Integrated Vector Basis.
Keywords: ZJ Transform, Linear Transformation, Matrix, Matrix Associated with the Linear Transformation, Integral Transform, Linear Algebra, Vector Spaces.

Introduction

In the analysis of dynamical systems governed by ordinary differential equations (ODEs) or in signal processing, integral transformations play a fundamental role in mapping time-domain problems to an algebraic frequency space. Traditionally, obtaining these transformations requires the explicit evaluation of improper integrals using analytical techniques of infinitesimal calculus—such as repeated integration by parts or residues in the complex plane—a process that often becomes algebraically dense or intractable as the complexity of the coupled function increases.

Through this approach, infinitesimal calculus is formally transformed into a problem of pure linear algebra. Thus, obtaining the integral transform is reduced to constructing a Differentiation Matrix ($[D]_B$), calculating its inverse operator ($[D]_B^{-1}$) as the integrating engine, and performing an algebraic evaluation on the boundary conditions of the domain.

Methodology: Comparison of Methods

1: Definition of Space and Input Elements

Traditional Method (Infinitesimal Calculus): The function f(t) is taken in isolation and the improper integral of the operator subject to the outer kernel is formulated:

$$ZJ[f(t)] = \frac{\beta^n}{z}\int_0^\infty e^{-\frac{z}{\beta}t} f(t)dt$$

The Matrix Method: The set of functions generated by the interaction between the signal and the kernel under the derivative operator is identified. With these, a vector subspace is defined and a v_N**Structured Basis with Integrated Kernel (B)** is constructed :

$$B = \left\{v_1(t)e^{-\frac{z}{\beta}t}, v_2(t)e^{-\frac{z}{\beta}t}, \dots, v_N(t)e^{-\frac{z}{\beta}t}\right\}$$

The function to be transformed is uniquely mapped as a vector of initial coordinates

2: Change Operator Execution

Traditional Method: The integration by parts algorithm (u, dv) is executed iteratively over the entire product of functions.

Matrix Method: The ordinary derivative operator d/ dtis applied. *only* the basic elements to construct the **Derivation Matrix [D]B** .Immediately, through algebraic matrix operations, its inverse is obtained:

$$[D]_B^{-1} = inv([D])_B$$

This inverse matrix $[D]_B^{-1}$operates as the **exact integration operator** of the entire space simultaneously.

3: Core Coupling

Traditional Method: The transformed domain variables (z, b or the substituted frequency $\alpha= \frac{z}{\beta}$) emerge in a scattered manner after evaluating the algebraic limits of each of the integrated parts.

Matrix Method: The frequency variables (z, b) are trapped and ordered from the previous step within the algebraic coefficients of the inverse matrix itself $[D]_B^{-1}$.

4: Boundary Assessment and Solution

Traditional Method: The analytical limits of the integrated function from 0 to infinity are evaluated, requiring a rigorous convergence analysis to ensure that the upper limit tends to zero.

Matrix Method: Multiplying the inverse matrix by the original coordinates, we obtain the integrated coordinate vector C_{int} =$[D]_B^{-1}*\ C_f$Because the exponential kernel extinguishes the base at infinity, the final step reduces to a **direct arithmetic evaluation on the lower boundary (t=0)**:

$$ZJ[f(t)] = \frac{\beta^n}{z} * \ (-c_{int}\ t = 0)$$

Let's prove that it is an isomorphism

Let's now see that the Transformation is an Isomorphism, it preserves information, let's demonstrate:

Thus, the linear transformation D from T: V to V must be injective and surjective, covering the entire space one by one, where V is a finite-dimensional vector space.

$$B = \left\{v_1(t)e^{-\frac{z}{\beta}t}, v_2(t)e^{-\frac{z}{\beta}t}, \dots, v_N(t)e^{-\frac{z}{\beta}t}\right\}$$

With $|[D]_B| \neq 0$, it must have the kernel $\{D\} = \{0\}$as the exponential function of the transformation is introduced when differentiating we eliminate isolated constants but the exponential remains so no non-trivial combination of the base B will give 0 then we have:

$$[D]_B * v_B = 0$$

It has to be $v_B = 0$this way, the operator is injective, now if it is surjective, if the nullity or of the Range Theorem we have the following:

$$dim(kernel\ [D]_B) +\ dim(imagen\ [D]_B) = dim\ V$$
$$0 +\ dim(imagen\ [D]_B) = dim\ V$$
$$dim(imagen\ [D]_B) = dim\ V$$

They have the same size, so it is Bijective and therefore an Isomorphism.

This tells us that the operator D with the kernel inside is of the existence of the inverse matrix, $[D]_B^{-1}$so the generating matrix of the transformation is$M =\ [T]_B *\ [D]_B^{-1}$

Commutation Theorem

In the context of linear algebra in finite-dimensional spaces, the theorem establishes the relationship between a differential operator in the time domain and its matrix counterpart, acting on the coordinate vector.

Linear Representation Theorem (Operator Isomorphism):

be V_Na finite-dimensional vector space over a field {K}, and let B = {v1, v2, …, vN} be an ordered basis of V_NN. If D: V_Na $V_N$is a linear operator (like the derivative operator) and $[D]_B$is its matrix representation with respect to the basis B, then for any functional vector f belonging to N $V_N$with coordinate vector f_B, it holds that:

$$[D(f)]_B = [D]_B *\ [f]_B$$

That is, the application of the analytical operator D on the function commutes perfectly with the multiplication of the matrix $[D]_B$by the coordinate vector.

Demonstration

Since $f(t) \in V_N$it $B = \{v_1(t), v_2(t), \dots, v_N(t)\}$is a basis of space, $f(t)$it can be uniquely written as a linear combination of the basis elements:

$$f(t) = \sum_{j=1}^{N} c_j v_j(t)$$

Where the coordinate vector $f_B =\ [c_1, c_2, \dots, c_N]^T$applying the derivative operator D = d/dt to both sides of the equation and using the linearity property of the operator we have

$$D\big((f)\big) = D\left(\sum_{j=1}^{N} c_j v_j(t)\right) = \sum_{j=1}^{N} c_j D(v_j(t))$$

By definition of a linear transformation, the derivative of each element of the basis $D(v_j(t))$must be expressible as a linear combination of the same basis B. Now we put the coefficients of this transformation as$d_{ij}\ talque$

$$D(v_j(t)) = \sum_{j=1}^{N} d_{ij}(v_j(t))$$

Now, substituting in the above, we have

$$D\big((f)\big) = \sum_{j=1}^{N} c_j \left(\sum_{j=1}^{N} d_{ij}(v_j(t)) \right)$$
$$D\big((f)\big) = \sum_{j=1}^{N} \left(\sum_{j=1}^{N} d_{ij}\, c_j \right) (v_j(t))$$

Now we have $\big[D\big((f)\big)\big]_B = [D]_B * [f]_B$, multiplying with, $[D]_B^{-1} * \big[D\big((f)\big)\big]_B = [D]_B^{-1} * [D]_B * [f]_B$we have$[D]_B^{-1} * \big[D\big((f)\big)\big]_B = [f]_B$

This formally demonstrates that the inverse matrix $[D]_B^{-1}$is the only algebraic operator capable of reversing analytic differentiation in coordinate space, acting as the antiderivative .

Simultaneous Diagonalization and Representation in the Basis of Eigenfunctions

By employing a coupled basis with the exponential kernel of the transformation of the type:

$$B = \left\{ v_1(t) e^{-\frac{z}{\beta}t}, v_2(t) e^{-\frac{z}{\beta}t}, \dots, v_N(t) e^{-\frac{z}{\beta}t} \right\}$$

The associated derivation matrix [D]_B does not adopt an elementary diagonal form as seen in the examples; this is due to the differential couplings introduced by the product rule when operating on polynomial, trigonometric, or composite terms, forcing [D]_B to be like an upper triangular or Jordan matrix .

When a linear operator in a dimension space finite does not possess enough vectors own linearly independent to be completely diagonalizable, cannot to represent through a matrix strictly diagonal. Jordan's theorem states that always can to represent in its simplest, almost diagonal form, known like **Jordan's Canonical** Form .

The eigenvalue (self-value) λ repeated along the main diagonal. Some distributed along the superdiagonal immediate and zeros in all the others positions.

However, the diagonalization property of the differential operator D = {d/ dt } is based on the existence of an alternative basis C composed entirely of the operator's eigenfunctions. According to classical linear operator theory, a transformation T: V to V in a finite-dimensional space is diagonalizable if and only if there exists a basis C for V such that the associated matrix $[T]_C$is a diagonal matrix.

For the functional space analyzed here, such a basis exists (extending, if necessary, to the domain of complex variables) and is denoted as

$$C = \{ e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t} \}$$

That is, taking in its exponential form in the case of example 1 only exp(-zt /b), in the case of the trigonometric functions example their complex exponential parts, and extracting the function exp(-zt/b), where each element individually satisfies the eigenvalue equation of each example. Under this ideal basis, the differentiation matrix $[D]_C$manifests itself in a strictly diagonal form:

$$[D]_C = \begin{bmatrix} \lambda_1 & 0 \dots & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 \dots & \lambda_n \end{bmatrix}$$

The formal transition between the system's working matrix and its $[D]_B$diagonalized counterpart $[D]_C$It is governed by a similarity transformation using a change-of-basis matrix P, defined by the relation:

$$[D]_C = P^{-1} * [D]_B * P$$

Since the integral transform operator T acts on this same finite algebraic space and preserves the linear independence of the states, its matrix representation $[T]_C$ also adopts a diagonal structure.

By interacting on the same set of eigenfunctions, it is shown that both operators commute with each other (DT = TD) and, consequently, are simultaneously diagonalizable. This structural consistency validates the proposed isomorphism and guarantees that, even when operating directly on the coupled real basis $[D]_B$, the existence of its inverse $[D]_B^{-1}$ acts as an exact and unambiguous integration in the coordinate space.

How to choose the base

Looking for a closed vector **subspace under the derivative operator** so with any function f(t) that one wishes to transform, is to analyze the form of the function when it is differentiated consecutively.

Differential Closure

For a function f(t) to be transformable using the matrix method, it must belong to the class of functions whose successive derivatives can be expressed as a linear combination of a finite number of functions. Mathematically, the space V_N spanned by the basis B must satisfy the **closure condition**:

Yeah$v \in V_N so \frac{dv}{dt} \in V_N$

Some examples of basic bases

Polynomial functions $f(t) = t^m$ thus containing all descending degrees

$$B = \left\{t^m e^{-\frac{z}{\beta}t}, t^{m-1} e^{-\frac{z}{\beta}t}, \dots, te^{-\frac{z}{\beta}t}, e^{-\frac{z}{\beta}t}\right\}$$

Exponential functions$f(t) = e^{\alpha t}$

$$B = \left\{e^{\alpha t} e^{-\frac{z}{\beta}t}\right\}$$

Trigonometric functions$f(t) = \cos(bt)\ osen(bt)$

$$B = \left\{\cos(bt) e^{-\frac{z}{\beta}t}, sen(bt) e^{-\frac{z}{\beta}t}\right\}$$

Hyperbolic trigonometric functions$f(t) = \cosh(bt)\ osenh(bt)$

$$B = \left\{\cosh(bt) e^{-\frac{z}{\beta}t}, senh(bt) e^{-\frac{z}{\beta}t}\right\}$$

Example with a composite function such as $f(t) = t^2 sen(at)$the combination are 6 and are multiplied by the exponential kernel and thus you have the base to work with.

$$B = \left\{t^2 sen(at)\, e^{-\frac{z}{\beta}t}, t^2 cos(at)\, e^{-\frac{z}{\beta}t}, tcos(at)\, e^{-\frac{z}{\beta}t}, tsen(at)\, e^{-\frac{z}{\beta}t}, cos(at)\, e^{-\frac{z}{\beta}t}, sen(at)\, e^{-\frac{z}{\beta}t}\right\}$$

Application Examples:

Example 1 we will find the transformation of f(t) = 1 and f(t) = t without integrating directly.

$$ZJ[f(t)] = \frac{\beta^n}{z} \int_0^\infty e^{-\frac{z}{\beta}t} f(t) dt$$

The basis is

$$B = \left\{e^{-\frac{z}{\beta}t}, te^{-\frac{z}{\beta}t}\right\}$$

We apply the D operator

$\boldsymbol{D}\left(e^{-\frac{z}{\beta}t}\right) = -\frac{z}{\beta} e^{-\frac{z}{\beta}t}$, $\boldsymbol{D}\left(te^{-\frac{z}{\beta}t}\right) = -\frac{z}{\beta} te^{-\frac{z}{\beta}t} + e^{-\frac{z}{\beta}t}$putting it in column for Matrix A, $A = [D]_B$we have

$[D]_B = \begin{bmatrix} -\frac{z}{\beta} & 1 \\ 0 & -\frac{z}{\beta} \end{bmatrix}$Now we obtain its inverse, and it is $[D]_B^{-1} = \begin{bmatrix} -\frac{\beta}{z} & -\frac{\beta^2}{z^2} \\ 0 & -\frac{\beta}{z} \end{bmatrix}$now, as in this step, that the limits within the integral are evaluated, the signs are changed, it is evaluated at t = 0 and multiplied by the scale factor outside the ZJ transformation, which is $\frac{\beta^n}{z}$thus

$$= \begin{bmatrix} \frac{\beta^{n+1}}{z^2} & \frac{\beta^{n+2}}{z^3} \\ 0 & \frac{\beta^{n+1}}{z^2} \end{bmatrix}$$

Now the vector Vn which is required either v1(1,0) and v2(0,1) of the basis is given that

$$\boldsymbol{A} = \begin{bmatrix} \frac{\beta^{n+1}}{z^2} & \frac{\beta^{n+2}}{z^3} \\ 0 & \frac{\beta^{n+1}}{z^2} \end{bmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \frac{\beta^{n+1}}{z^2} \\ 0 \end{pmatrix} \textbf{and} \boldsymbol{A} = \begin{bmatrix} \frac{\beta^{n+1}}{z^2} & \frac{\beta^{n+2}}{z^3} \\ 0 & \frac{\beta^{n+1}}{z^2} \end{bmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \frac{\beta^{n+2}}{z^3} \\ \frac{\beta^{n+1}}{z^2} \end{pmatrix}$$

Table 1 transformations for some basic functions

Functión	ZJ Transform
1	$\frac{\beta^{n+1}}{z^2}$
t	$\frac{\beta^{n+2}}{z^3}$

Figure 1

And so, finally, integration is achieved in a manageable way.

Example 2 Let the transformation of f(t) = t^2 without direct integration be now.

The basis is

$$B = \left\{ e^{-\frac{z}{\beta}t}, te^{-\frac{z}{\beta}t}, t^2 e^{-\frac{z}{\beta}t} \right\}$$

We apply the D operator

$\boldsymbol{D}\left(e^{-\frac{z}{\beta}t}\right) = -\frac{z}{\beta} e^{-\frac{z}{\beta}t}, \boldsymbol{D}\left(te^{-\frac{z}{\beta}t}\right) = -\frac{z}{\beta} te^{-\frac{z}{\beta}t} + e^{-\frac{z}{\beta}t}, \boldsymbol{D}\left(t^2 e^{-\frac{z}{\beta}t}\right) = -\frac{z}{\beta} t^2 e^{-\frac{z}{\beta}t} + 2te^{-\frac{z}{\beta}t}$ now putting in column for Matrix A, in the positions $A = [D]_B$ we have

$$[D]_B = \begin{bmatrix} -\frac{z}{\beta} & 1 & 0 \\ 1 & -\frac{z}{\beta} & 2 \\ 0 & 0 & -\frac{z}{\beta} \end{bmatrix}$$

Now we obtain its inverse, and it is $[D]_B^{-1} = \begin{bmatrix} -\frac{\beta}{z} & -\frac{\beta^2}{z^2} & -\frac{2\beta^3}{z^3} \\ 0 & -\frac{\beta}{z} & -\frac{2\beta^2}{z^2} \\ 0 & 0 & -\frac{\beta}{z} \end{bmatrix}$ now, as in this step, that the limits within the integral are evaluated, the signs are changed, it is evaluated at t = 0 and multiplied by the scale factor outside the ZJ transformation, which is $\frac{\beta^n}{z}$ thus

$$[D]_B^{-1} = \begin{bmatrix} \frac{\beta^{n+1}}{z^2} & \frac{\beta^{n+2}}{z^3} & \frac{2\beta^{n+3}}{z^4} \\ 0 & \frac{\beta^{n+1}}{z^2} & \frac{2\beta^{n+2}}{z^3} \\ 0 & 0 & \frac{\beta^{n+1}}{z^2} \end{bmatrix}$$

Now the vector Vn which is required is v3(0,0,1) from the basis we have that

$$\boldsymbol{A} = \begin{bmatrix} \frac{\beta^{n+1}}{z^2} & \frac{\beta^{n+2}}{z^3} & \frac{2\beta^{n+3}}{z^4} \\ 0 & \frac{\beta^{n+1}}{z^2} & \frac{2\beta^{n+2}}{z^3} \\ 0 & 0 & \frac{\beta^{n+1}}{z^2} \end{bmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \frac{2\beta^{n+3}}{z^4} \\ \frac{2\beta^{n+2}}{z^3} \\ \frac{\beta^{n+1}}{z^2} \end{pmatrix}$$

Figure 2

t^2	$\frac{2\beta^{n+3}}{z^4}$

You can see that other components and possible integrations are appearing again.

We can then generalize the following for the function $f(t) = t^k$for k>0 in this base space

$$\boldsymbol{I}[\boldsymbol{t}^k] = \frac{\boldsymbol{\beta}^n}{\boldsymbol{z}}\left([\boldsymbol{D}]_{\boldsymbol{B}}^{-1} * [\boldsymbol{t}^k]_{\boldsymbol{B}}\right)$$

With a base like

$$\boldsymbol{B} = \left\{\boldsymbol{e}^{-\frac{z}{\beta}t}, \dots, \boldsymbol{t}^k\boldsymbol{e}^{-\frac{z}{\beta}t}\right\}$$

We can obtain the following expression

$$\boldsymbol{I}[\boldsymbol{t}^k] = \left(\sum_{i=0}^{k} \frac{\boldsymbol{k}!}{(\boldsymbol{k}-\boldsymbol{i})!} * \frac{\boldsymbol{\beta}^{n+i+1}}{\boldsymbol{z}^{i+2}} * \boldsymbol{t}^{k-i}\right)\boldsymbol{e}^{-\frac{zt}{\beta}}$$

Let k = 3

$$\boldsymbol{I}[\boldsymbol{t}^3] = \left(\sum_{i=0}^{3} \frac{\mathbf{3}!}{(\mathbf{3}-\boldsymbol{i})!} * \frac{\boldsymbol{\beta}^{n+i+1}}{\boldsymbol{z}^{i+2}} * \boldsymbol{t}^{3-i}\right)\boldsymbol{e}^{-\frac{zt}{\beta}} = \frac{\mathbf{6}\boldsymbol{\beta}^{n+4}}{\boldsymbol{z}^5}$$

Example 3 Trigonometric functions, let's put a here $=-\frac{z}{\beta}$

$$B = \left\{\cos(bt)e^{-\frac{z}{\beta}t}, sen(bt)e^{-\frac{z}{\beta}t}\right\}$$

Applying the D operator

$\boldsymbol{D}(e^{at}\cos(bt)) = a(e^{at}\cos(bt)) - b(e^{at}\operatorname{sen}(bt))$is (a, -b)
$\boldsymbol{D}(e^{at}\operatorname{sen}(bt)) = b(e^{at}\cos(bt)) + a(e^{at}\operatorname{sen}(bt))$is (b,a)

Putting in column for Matrix A, in the positions $A = [D]_B$we have

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Now we obtain its inverse and it is $[D]_B^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$in terms of **a** $=-\frac{z}{\beta}$

$$[D]_B^{-1} = \frac{\beta^2}{z^2+b^2\beta^2}\begin{bmatrix} -\frac{z}{\beta} & -b \\ b & -\frac{z}{\beta} \end{bmatrix} = \frac{\beta}{z^2+b^2\beta^2}\begin{bmatrix} -z & -b\beta \\ b\beta & -z \end{bmatrix}$$

Let's integrate the function

$ZJ[(\cos(bt)] = \frac{\beta^n}{z}\int_0^\infty e^{-\frac{z}{\beta}t}(\cos(bt))dtandZJ[(\operatorname{sen}(bt)] = \frac{\beta^n}{z}\int_0^\infty e^{-\frac{z}{\beta}t}(\operatorname{sen}(bt))dt$

Now we are interested in the position of the base v1 or (1,0), we apply the limit at t=0 and by the scale factor

$$= [D]_B^{-1} * \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{z\beta}{z^2+b^2\beta^2} \\ \frac{b\beta^2}{z^2+b^2\beta^2} \end{pmatrix} * \frac{\beta^n}{z} = \begin{pmatrix} -\frac{\beta^{n+1}}{z^2+b^2\beta^2} \\ -\frac{b\beta^{n+2}}{z(z^2+b^2\beta^2)} \end{pmatrix}$$

With the base position v1 or (0,1)

$$= [D]_B^{-1} * \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\dfrac{b\beta^2}{z^2+b^2\beta^2} \\ -\dfrac{bz\beta}{z^2+b^2\beta^2} \end{pmatrix} * \frac{\beta^n}{z} = \begin{pmatrix} \dfrac{b\beta^{n+2}}{z(z^2+b^2\beta^2)} \\ -\dfrac{b\beta^{n+1}}{z^2+b^2\beta^2} \end{pmatrix}$$

Having the two integrations

$$ZJ[(\cos(bt)] = \frac{\beta^{n+1}}{z^2+b^2\beta^2}$$

$$ZJ[(\mathrm{sen}(bt)] = \frac{b\beta^{n+2}}{z(z^2+b^2\beta^2)}$$

Example 4 Hyperbolic trigonometric functions, let's put a here $=-\frac{z}{\beta}$

$$B = \left\{\cosh(bt)e^{-\frac{z}{\beta}t}, senh(bt)e^{-\frac{z}{\beta}t}\right\}$$

Applying the D operator

$\boldsymbol{D}(e^{at}\cosh(bt)) = a(e^{at}\cosh(bt)) + b(e^{at}\mathrm{senh}(bt))$is (a,b)
$\boldsymbol{D}(e^{at}\mathrm{senh}\,(bt)) = b(e^{at}\cosh(bt)) + a(e^{at}\mathrm{senh}\,(bt))$is (b,a)

Putting in column for Matrix A, in the positions $A = [D]_B$we have

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Now we obtain its inverse and it is $[D]_B^{-1} = \frac{1}{a^2-b^2}\begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$in terms of **a** $=-\frac{z}{\beta}$

$$[D]_B^{-1} = \frac{\beta^2}{z^2-b^2\beta^2}\begin{bmatrix} -\dfrac{z}{\beta} & -b \\ -b & -\dfrac{z}{\beta} \end{bmatrix} = \frac{\beta}{z^2-b^2\beta^2}\begin{bmatrix} -z & -b\beta \\ -b\beta & -z \end{bmatrix}$$

Let's integrate the function

$ZJ[(\cosh(bt)] = \frac{\beta^n}{z}\int_0^\infty e^{-\frac{z}{\beta}t}(\cosh(bt))dtandZJ[(\mathrm{senh}(bt)] = \frac{\beta^n}{z}\int_0^\infty e^{-\frac{z}{\beta}t}(\mathrm{senh}(bt))dt$

Now we are interested in the position of the base v1 or (1,0), we apply the limit at t=0 and by the scale factor the other integration also comes out automatically.

$$= [D]_B^{-1} * \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\dfrac{z\beta}{z^2-b^2\beta^2} \\ -\dfrac{b\beta^2}{z^2-b^2\beta^2} \end{pmatrix} * \frac{\beta^n}{z} = \begin{pmatrix} \dfrac{\beta^{n+1}}{z^2-b^2\beta^2} \\ \dfrac{b\beta^{n+2}}{z(z^2-b^2\beta^2)} \end{pmatrix}$$

$$ZJ[(\cosh(bt)] = \frac{\beta^{n+1}}{z^2-b^2\beta^2}$$

$$ZJ[(\mathrm{senh}(bt)] = \frac{b\beta^{n+2}}{z(z^2-b^2\beta^2)}$$

Example 5 Composite function with $f(t) = tcos(at)$the combination are 4 and are multiplied by the exponential kernel, **here the** $\boldsymbol{\alpha}=\frac{z}{\beta}$

$$B = \left\{tcos(at)e^{-\frac{z}{\beta}t}, tsen(at)e^{-\frac{z}{\beta}t}, cos(at)e^{-\frac{z}{\beta}t}, sen(at)e^{-\frac{z}{\beta}t}\right\}$$

Applying the D operator

$$\boldsymbol{D}(te^{-\alpha t}\cos(at)) = e^{-\alpha t}\cos(at) - ate^{-\alpha t}\,\mathrm{sen}(at) - \alpha te^{-\alpha t}\cos(at)$$
$$\boldsymbol{D}(te^{-\alpha t}\mathrm{sen}(at)) = e^{-\alpha t}\mathrm{sen(at)} + ate^{-\alpha t}\cos(at) - \alpha te^{-\alpha t}\mathrm{sen}(at)$$
$$\boldsymbol{D}(e^{-\alpha t}\cos(at)) = -ae^{-\alpha t}\,\mathrm{sen}(at) - \alpha e^{-\alpha t}\cos(at)$$
$$\boldsymbol{D}(e^{-\alpha t}\mathrm{sen}(at)) = ae^{-\alpha t}\cos(at) - \alpha e^{-\alpha t}\mathrm{sen}(at)$$

Putting in column for Matrix A, in the positions $A = [D]_B$we have

$$A=\begin{bmatrix}-\alpha & a & 0 & 0\\ -a & -\alpha & 0 & 0\\ 1 & 0 & -\alpha & a\\ 0 & 1 & -a & -\alpha\end{bmatrix}$$

Now we obtain its inverse and it is

$$[D]_B^{-1}=\frac{1}{(\alpha^2+a^2)^2}\begin{bmatrix}-\alpha(\alpha^2+a^2) & -a(\alpha^2+a^2) & 0 & 0\\ a(\alpha^2+a^2) & -\alpha(\alpha^2+a^2) & 0 & 0\\ a^2-\alpha^2 & -2a\alpha & -\alpha(\alpha^2+a^2) & -a(\alpha^2+a^2)\\ 2a\alpha & a^2-\alpha^2 & a(\alpha^2+a^2) & -\alpha(\alpha^2+a^2)\end{bmatrix}$$

Now that we know what the transform will be, when applying the limit of ordinary integration at infinity, all of them go to 0, but when applying t=0 and inverting the sign, only $cos(at)e^{-\frac{z}{\beta}t}$ the third position remains, so $V_N$now $v_B=\{1,0,0,0\}$we are looking for the transform of that position.

$$=[D]_B^{-1}*\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}\frac{-\alpha}{(\alpha^2+a^2)}\\ \frac{a}{(\alpha^2+a^2)}\\ \frac{a^2-\alpha^2}{(\alpha^2+a^2)^2}\\ \frac{-2a\alpha}{(\alpha^2+a^2)^2}\end{pmatrix}*\frac{\beta^n}{z}=\left(\frac{\alpha^2-a^2}{(\alpha^2+a^2)^2}\right)*\frac{\beta^n}{z}$$

We still have the desired integration .

Example 6 Dirac Delta Function with $f(t)=\delta(t-t_0)$we define the basis with the impulse function and the response by differentiating:

$$B=\left\{\delta(t-t_0)e^{-\frac{z}{\beta}t},H(t-t_0)e^{-\frac{z}{\beta}t}\right\}$$

Where H is the Heaviside function https://es.wikipedia.org/wiki/Funcion_escalon_de_Heaviside The Dirac delta function is the Heaviside derivative . And remember, position is$v_B=\{1,0\}$

Applying the D operator

$$\boldsymbol{D}\left(\delta(t-t_0)e^{-\frac{z}{\beta}t}\right)=\delta(t-t_0)'e^{-\frac{z}{\beta}t}-\frac{z}{\beta}\delta(t-t_0)e^{-\frac{z}{\beta}t}$$

$$\boldsymbol{D}\left(H(t-t_0)e^{-\frac{z}{\beta}t}\right)=H(t-t_0)'e^{-\frac{z}{\beta}t}-\frac{z}{\beta}H(t-t_0)e^{-\frac{z}{\beta}t}$$

Putting in column for Matrix A, in the positions $A=[D]_B$we have

$$A=\begin{bmatrix}-\frac{z}{\beta} & 1\\ 0 & -\frac{z}{\beta}\end{bmatrix}$$

Now the reverse

$$[D]_B^{-1}=\begin{bmatrix}-\frac{\beta}{z} & -\frac{\beta^2}{z^2}\\ 0 & -\frac{\beta}{z}\end{bmatrix}*\frac{\beta^n}{z}=\begin{bmatrix}-\frac{\beta^{n+1}}{z^2} & -\frac{\beta^{n+2}}{z^3}\\ 0 & -\frac{\beta^{n+1}}{z^2}\end{bmatrix}*\begin{pmatrix}1\\0\end{pmatrix}=\begin{bmatrix}-\frac{\beta^{n+1}}{z^2}\\ 0\end{bmatrix}$$

When evaluating at t=0, the sign changes, and thus, the same result is obtained as using integration.

$$\boldsymbol{ZJ}\left[\delta(t-t_0)e^{-\frac{z}{\beta}t}\right]=\frac{\boldsymbol{\beta^{n+1}}}{\boldsymbol{z^2}}\boldsymbol{e^{-\frac{z}{\beta}t_0}}$$

Conclusions

Among the conclusions we can see how this method within linear algebra with Linear Transformations and associated matrix can be used with the **ZJ Transformation** and thus we see how certain functions can obtain their transformations without needing to do the integral, numerically this methodology can be associated, this leads us to be able to apply this same thing to other similar transforms such as Laplace, Fourier, Z, Hankel, Mellin, Elzaki or Sumudu.

The kernel of the **Laplace transform** structurally coincides with the kernel used in this work; obtaining the Laplace transform for polynomial, harmonic, or exponential functions reduces to exactly the same matrix inversion problem. In the case of the **Fourier transform**, whose kernel is the complex exponential, the method is applied identically, expanding the analysis to the field of complex numbers. As demonstrated in the harmonic analyses, the presence of pure frequencies eliminates real couplings and leads to spectral representations using complex Jordan blocks, allowing the exact transform to be evaluated without resorting to improper integrals at infinity. For the **Z-transform**, which is the discrete counterpart of the Laplace transform, the method is generalized in this case by replacing the continuous differential operator with the discrete shift operator.

Elzaki, Sumudu, and Abdo transformations (and others such as the Mohand, Kamal, and Shehu transformations, etc.) belong to the same mathematical family, where they are generalizations or scaling variations of the Laplace transform. All of them, including the **ZJ transform,** share a generic integral structure that follows the form:

$$T[f(t)] = K(s)\int_0^{\infty} e^{-p(s)t} f(t)dt$$

Where K(s) is an external scaling factor and p(s) is the kernel decay frequency. Since **they all share the exponential kernel**, the vector space V_N and the coupled basis B constructed in the method are **identical** for all of them.

The derivation matrix $[D]_B$ and their Jordan blocks will have exactly the same shape as in your Examples 1 to 6. The only difference will lie in the simple **algebraic substitution of variables** and in the factor that multiplies the inverse matrix when evaluating the boundary at t=0.

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