International Journal of Latest Research in Engineering and Technology (IJLRET)
ISSN: 2454-5031
www.ijlret.com || Volume 12 - Issue 11 || November 2026 || PP. 19-29

Examples of the pollutant source estimation inverse problem using the SVD/Tikhonov method

M. Sc. Zenteno Jiménez José Roberto
Geophysical Engineering, National Polytechnic Institute, Mexico City
ESIA- Ticóman Unit Mayor Gustavo A. Madero
Email: jzenteno@ipn.mx

Abstract: This paper presents a **method for estimating the intensity (mass emitted) of a point source of air pollution** , based on the methodology of Dr. David Parra Guevara and Yuri N. Skiba, which releases the pollutant in the form of **N successive instantaneous emissions (** Dirac delta pulses) at known times. The problem is then generalized to multiple pulses.The estimate is based on a **time series of concentration measured at a monitoring site** , combined with a well-formulated atmospheric dispersion model. The estimation problem is formulated as an **unconstrained quadratic optimization problem** , whose optimal solution reduces to solving a **positive (positive definite) system of linear equations of size N×N using the SVD method and** Tikhonov regulation .

Introduction

The estimate is based on a **time series of concentration measured at a monitoring site** , combined with a well-formulated atmospheric dispersion model (in the Hadamard sense). The estimation problem is formulated as an **unconstrained quadratic optimization problem** , whose optimal solution reduces to solving a **positive (positive definite) system of linear equations of size N×N** .

The emission rate of a pollution source is fundamental information for environmental research; air quality models require it as input data to forecast pollution levels and assess risks to the population. However, **estimating emission rates is, in general, a poorly defined problem** , lacking a universal solution method; each method must introduce system-specific simplifications.

This type of methodology is particularly relevant in **emergency scenarios and in response to industrial or nuclear accidents** , such as nuclear plant accidents, where concentration measurements are available, but it is unknown how much material was actually emitted and at what times.

Positive Systems of Linear Equations

We have a matrix A defined as positive of nxn, so Ax = b, with the A^T= A, its transpose matrix is the same as the original matrix ($A^T = A$)and $x^t Ax \geq 0$for all x belonging to R to the n, so A is a real symmetric and positive matrix, with the proven properties of [1] and then applying the SVD method.

Using the eigenvalues of a positive symmetric matrix of a positive definite real symmetric matrix, which are always strictly greater than zero real numbers, whose determinant and trace, the determinant of the matrix (the product of all its eigenvalues) and its trace (the sum of them) are both greater than zero, which obey the following properties which are demonstrated in the article on **Estimation of the intensity of the successive instantaneous emissions released by air pollution point source [1].**

Mathematical Model of Dispersion

The starting point is a linear three-dimensional advection -diffusion-reaction model that describes the transport of a passive primary pollutant (e.g. CO, SO2 , NOx or $_{soot}$) in a bounded domain $D = D \times (0, H) \subset \mathbb{R}^3$, with a boundary divided into a cylindrical lateral surface S, a base S_0, and a top cap SH. It is based on the following general model of [1]

The Transport Equation

$$\frac{\partial \varphi}{\partial t} + U \cdot \nabla\varphi + \sigma\varphi - \nabla \cdot (\mu \nabla\varphi) - \frac{\partial}{\partial z}\left(\mu_z \frac{\partial \varphi}{\partial z}\right) + \nabla \cdot \varphi^s = f(r,t)$$

where φ(r,t) is the concentration of the contaminant, $\boldsymbol{U}$ = (u,v,w) is the wind field (known, and satisfying the continuity equation ∇ ·U = 0), σ(r,t) ≥ 0 is the chemical transformation coefficient, μ(r,t) and μẑ (r,t) are the horizontal/vertical turbulent diffusion arrangements, and $\varphi^s = -v^s \varphi e_3$ represents the particle settling term with constant rate $v^s > 0$.

Initial and boundary conditions

The system is completed with an initial condition $\varphi(r,0) = \varphi^0(r) \geq 0$, and with boundary conditions that distinguish the inflow part S^- (where $U_n < 0$) and the outflow part S^+ (where $U_n \geq 0$):

- In S^-: total flux (diffusive + advective) zero — no source outside the domain is assumed.
- In S^+: diffusive flux is disregarded compared to advective flux (free outlet condition).
- In S_0 (base): there is no contaminant flow, $\mu \hat{} \nabla \varphi \cdot n = 0$.
- In SH (top): particle sedimentation is additionally incorporated.

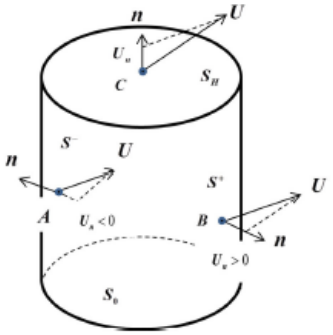


Figure 1 Boundary Conditions of the General Transport Equation Model
Source: Yuri N. Skiba , David Parra Guevara Mathematical Methods for the Assessment and Control of Industrial Emissions (Springer)

These conditions make the problem well-posed in the **Hadamard sense** (existence, uniqueness, and continuous dependence of the solution on the initial data and forcing), which is demonstrated by proving that the problem operator, $\boldsymbol{A}$, is non-negative:

$(A\varphi , \varphi) \geq 0 \Rightarrow \|\varphi\|_2 \leq T \cdot \max \|f (r,t)\|_2 + \|\varphi^0\|_2$

This bound guarantees the uniqueness of the solution and its continuous dependence on the initial conditions and the forcing, and it is the basis of all subsequent development. Additionally, integrating the transport equation over the domain yields a physically consistent **mass balance equation** : the total mass of the pollutant increases due to emission rates and decreases due to advective outflow, chemical transformation, and sedimentation in the soil.

For the numerical solution of the model, a balanced and absolutely stable second-order difference scheme, the splitting method, and Crank-Nicolson schemes are used (see reference [2]).

Algebraic foundation: positive systems of linear equations

Before presenting the estimation method, the article develops general results on linear systems $\Lambda x = b$ where the matrix Λ is **symmetric and positive definite** ("positive system"). These results—expressed as Properties 1–3, Lemma 1, and Theorem 1 with its Corollary—are then used to characterize the solution to the estimation problem (see reference [1]):

- Property 1: The eigenvalues of a real symmetric matrix are real, and its eigenvectors form an orthonormal basis of $\mathbb{R}^N$.
- Property 2: The eigenvalues of a positive definite matrix are all positive.
- Property 3: Every positive definite matrix is non-singular (invertible).
- Lemma 1: The unique solution of $\Lambda x = b$ can be written as a combination of the eigenvectors , $x = \Sigma (v_i^T b / \lambda_i) v_i$.

Starting from Lemma 1, **Theorem 1** establishes an explicit formula for the sensitivity of the solution to perturbations δb on the right-hand side:

$$\|\delta x\|^2 = \Sigma_i (v_i^T \delta b / \lambda_i)^2 \text{ and its Corollary: } \| \delta x \| \leq (\sqrt{N} / \lambda_1) \| \delta b \|$$

This last inequality indicates that the system is **well-conditioned** when the ratio $\sqrt{N}/\lambda_1$ is not large, that is, when the smallest eigenvalue λ_1 of the matrix is not excessively small. This criterion will be revisited later to choose the regularization parameter ε of the method.

Proposed method for estimating the intensity of the source

The point source is assumed to emit the pollutant as a superposition of N instantaneous emissions at a known point r_0 and at known times t_i:

$$Q(t) = \Sigma_{i=1}^N Q_i \, \delta(t - t_i)$$

Dirac delta function and the coefficients Q_i (the intensities) are the unknowns of the problem. A time series of the measured concentration, φ d (R, τ_k), at a monitoring point R ∈ D is assumed to be known.

Now the non-homogeneous Advection-Diffusion Partial Differential Equation: Non-homogeneous will be used for the Examples, in one direction

$$\frac{\partial \emptyset}{\partial t} + u\frac{\partial \emptyset}{\partial x} - \mu\frac{\partial^2 \emptyset}{\partial x^2} + \sigma\emptyset = \sum_{i=1}^{5} Q_i \delta(t - t_i)\delta(x - x_0)$$

With

$$\mu\frac{\partial \emptyset(0,t)}{\partial x} - u\emptyset(0,t) = 0$$
$$\mu\frac{\partial \emptyset(l,t)}{\partial x} = 0$$
$$\emptyset(x,0) = 0$$

Linear decomposition of the solution

Since the scattering model is linear, its solution can be written as a superposition of three independent contributions:

$$\varphi(\, r,t\,) = C_0(\, r,t\,) + C(\, r,t\,) + \Sigma_{i=1}{}^{N}\, Q_i\, C_i(\, r,t\,)$$

- is, the system's response to $\delta(t - t_i)\delta\,(r - r_0)$.
- C_0: response due solely to the initial distribution of the contaminant $\varphi^0(r)$.
- C: response due to known sources f(r,t) already present in the domain.

C_i: "influence function" or fundamental solution associated with a unit pulse emitted at r_0 at time t_i — that The point is that **C_0, C, and each C_i can be calculated in advance** by solving the dispersion model independently of the (still unknown) values of Q_i. The estimation problem then reduces to finding the Q_i coefficients that best fit the superposition to the observed data.

Formulation as an optimization problem

Tikhonov -type regularization least squares):

$$\text{Min } \Phi = \tfrac{1}{2}\int_0{}^{T}\, [\, \varphi d(\, R,t\,) - C_0(\, R,t\,) - C(\, R,t\,) - \Sigma_j\, Q_j\, C_j(\, R,t\,)]^2\, dt + (\varepsilon/2)\Sigma_j\, Q_j{}^2$$

The first term penalizes the discrepancy between the model and the observed data; the second term, weighted by the **regularization parameter ε > 0** , penalizes the magnitude of the estimated intensities. This is illustrated by a scalar example: as ε → 0, the solution tends to fit the data exactly, while as ε → ∞, the solution tends to zero—hence its role in "controlling" the sensitivity of the problem.

Reduction to a positive linear system

The first-order condition ∇ Φ = 0 leads to a linear system of size N×N:

$$\Lambda Q = b, \text{ where } \Lambda = G + \varepsilon I$$

where ***G*** is the symmetric matrix with entries $G_i = \int_0{}^{T}\, C_i(\, R,t\,)C(\, R,t\,)\, dt$, *I* is the identity matrix, ***Q*** = $(Q_1,\ldots,Q_n)^T$ is the vector of intensities to be estimated, and ***b*** is the vector with components $b_i = \int_0{}^{T}\, [\, \varphi_d(\, R,t\,) - C_0(\, R,t\,) - C(\, R,t\,)]\, C_i(\, R,t\,)\, dt$.

Two central results are demonstrated (see reference [1]):

- **Theorem 2:** For all ε > 0, the matrix Λ = G + εI is positive definite (even though G is only positive semidefinite).
- **Theorem 3:** The solution of the linear system (27) is the global minimum of Φ, since its Hessian coincides with Λ, which is positive definite ⇒ Φ is strictly convex.

Thanks to these results, the solution Qεalways exists and is unique for any ε > 0, and —using Lemma 1 and Corollary 1 from the previous section— an explicit bound on the sensitivity of the solution to errors in the data is obtained:

$$Q\varepsilon = \Sigma_i\, (v_i{}^T b / (\lambda_i + \varepsilon))\, v_i \,\|\, \delta Q_\varepsilon \,\| \le (\sqrt{N} / (\lambda_1 + \varepsilon))\, \|\, \delta b \,\|$$

This expression is the most important practical contribution of the method: the **parameter ε allows controlling the conditioning of the problem** even when the smallest eigenvalue λ_1 of G is very small (or zero), avoiding the amplification of measurement errors by reversing the system.

Now for the Adjoint Model we have the Adjoint Advection-Diffusion Partial Differential Equation: Non-homogeneous (final conditions) as

$$-\frac{\partial \emptyset}{\partial t} - u\frac{\partial \emptyset}{\partial x} - \mu\frac{\partial^2 \emptyset}{\partial x^2} + \sigma\emptyset = \delta(x - x_i)\delta(t - t_i)$$

With

$$\mu\frac{\partial g(L,T)}{\partial x} + ug(L,T) = 0$$
$$\mu\frac{\partial g(0,T)}{\partial x} = 0$$
$$g(x,T) = 0$$

Alternative method SVD/ Tikhonov the Mathematics behind the inverse method.

The observed data with noise are entered as follows: $\emptyset_d = \emptyset_{exacta} + s * \max(\emptyset_{exacta}) * (rand(t) - 0.5)$, now $\emptyset_{exacta}$: the "true" concentration, generated with the actual Qi . Where s is the relative noise parameter (e.g., 0.05 = 5% of the peak maximum). With $(rand(t) - 0.5)$in [-0.5, 0.5] the uniform noise centered at or near zero. This will simulate that a real sensor never measures the exact concentration.

Construction of the Gij matrix and the bi vector

The vector Q that minimizes the squared error between the observed and reconstructed values is sought:

$$\Phi(Q) = \int_0^T [\varphi d(R,t) - C_0(R,t) - C(R,t) - \Sigma_j Q_j C_j(R,t)]^2 \, dt$$

This is a quadratic function of Q (as an N-dimensional paraboloid), so its minimum is found where the gradient is zero, like this (see reference [1]).

$$\frac{\partial \emptyset}{\partial Q_i} = 0$$

Thus, we finally have

$$\int_0^T \left[C_i(R,t)C_j(R,t) + \varepsilon I\right]dt\, Q_i = \int_0^T \left(\emptyset_d(t)C_i(R,t) - C_i(R,t)C_0(R,t) - C_i(R,t)C(R,t)\right)dt$$

With

$$\Lambda Q = b \text{ where } \Lambda = G + \varepsilon I$$

$$G = \int_0^T \left[C_i(R,t)C_j(R,t)\right]dt \text{ y } b = \int_0^T \left(\emptyset_d(t)C_i(R,t) - C_i(R,t)C_0(R,t) - C_i(R,t)C(R,t)\right)dt$$

Solution by SVD / Tikhonov

Since G is ill-conditioned (the eigenvalues are very small or almost zero), inverting it directly strongly amplifies the noise of $\emptyset_d(t)$, so instead of minimizing only $\| \varphi d(R,t) - C_0(R,t) - C(R,t) - \Sigma_j Q_j C_j(R,t) \|^2$ we minimize $\| \varphi d(R,t) - C_0(R,t) - C(R,t) - \Sigma_j Q_j C_j(R,t) \|^2 + \varepsilon \|\Sigma_j Q_j\|^2$

With solution $(G + \varepsilon I)Q\varepsilon = b \Rightarrow Q\varepsilon = (G + \varepsilon I)^{-1} b$, εas it raises the small values of the matrix G although all values are raised more or less by epsilon making the numerical inversion stable, the bias balance in the variance, then $\Lambda = G + \varepsilon I = U \sum V^T$, so the inv $(U \sum V^T)$= U $\Sigma^+ V^T$where Σ^+it inverts every singular value except the 0 values

What is done in the article (Lemma 1 + Theorems 1-3 (see reference [1]) **):**

Since G is symmetric and positive semidefinite (Section 3, Property 1-3 (see reference [1])), it factors:

$$G = V\Lambda V^{-1} = V\Lambda V^T$$

With V as orthogonal (the column = eigenvectors) and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots \lambda_N)$thus our solution $AQ_\varepsilon = b$with A = G + εI can be written as such

$$Q_\varepsilon = \Sigma_i (v_i^T b / (\lambda_i + \varepsilon))\, v_i$$

And also from there a sensitivity bound $\| \delta Q\varepsilon \| \leq (\sqrt{N} / (\lambda_1 + \varepsilon)) \| \delta b \|$ with λ_1 the smallest eigenvalue, so for a symmetric and positive semidefinite matrix like G, the SVD coincides with the spectral decomposition, the singular values are the eigenvalues and U = V then the pinv(A)= U $\Sigma^+ V^T$and so on $Q_\varepsilon = pinv(A) * b$, so the advantage is that the pinv automatically truncates the singular values and also operates if epsilon is 0

With error metrics

$$Errdatos = \frac{||\emptyset_d - \emptyset exacta||}{\emptyset exacta} \qquad ErrQ = \frac{||Q_\varepsilon - Qreal||}{Qreal}$$

Reconstruction and comparison

$$\emptyset reconstruida = \sum_{i=1}^{N} Q_{\varepsilon} C_i(R,t)$$

It is simply the direct model evaluated with the recovered rates, to be visually compared against phi_d (with noise) and phi_exacta (without noise) — if the method worked well, the red curve in the later examples should closely follow the dotted blue curve of the actual rates.

Numerical examples of validation

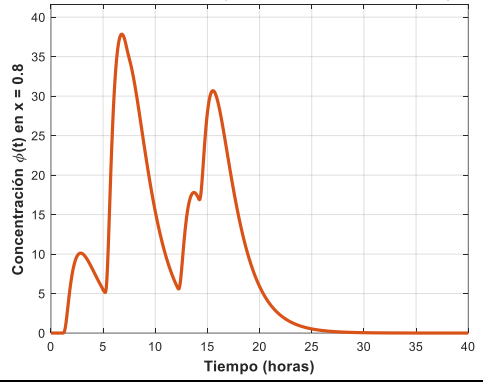
To demonstrate and recreate the method, the example from the original article will be used (see reference [1]), a version of the one-dimensional scattering model, with Robin /Neumann type boundary conditions at x=0 and x=L, and initial condition φ(x,0) = 0. The parameters used are:

Wind speed	**or**	**0.25**
Diffusion coefficient	**μ**	**0.05**
Transformation rate	**σ**	**0.001**
Domain length	**L**	**1**
Broadcast site	**x_0**	**0.2**
Monitoring site	**R**	**0.8**

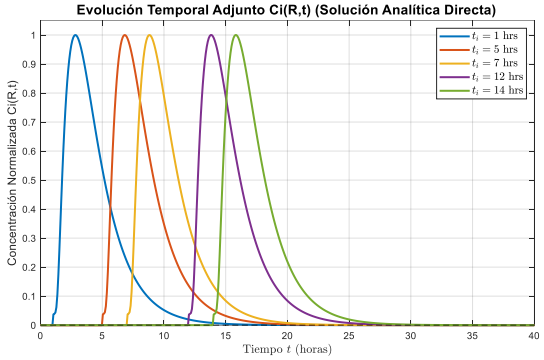
Five instantaneous emissions were set at times t_1=1, t_2=5, t_3=7, t_4=12 and t_5=14, with pre-established "true" intensities Q_1=10, Q_2=25, Q_3=3, Q_4=15 and Q_5=20. With these values, a synthetic concentration φ(R,t) was generated, and from it "observed" data φd(R,t) = φ(R,t) + δφ (R,t), where δφ is uniform random noise whose amplitude is controlled with a parameter s (s = 0, 0.05 and 0.1, i.e., 0%, ~3.8% and ~7.6% relative error in the data).

Result of the Direct Dispersion Method

Analytical Solution



Adjoint Weights – Adjoint Model



Investment Solution

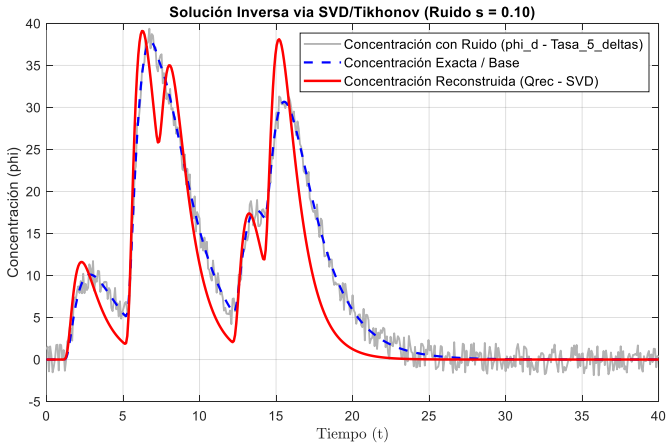


Figure 2: Solution of the Inverse Method via SVD/ Tikhonov

===== ========== RESULTS OF THE INVERSE METHOD (SVD) ==========

Eigenvalues of matrix G: 0.4718 0.4858 0.7164 0.9475 0.9791 Regularization parameter epsilon: 1e-06 Relative error in concentration data: 0.0790 (7.90%) Real Rates Q = [10, 35, 3, 15, 20];	Qepsilon Recovered Rates (via SVD): Q1 = 11.9058 Q2 = 38.8788 Q3 = 3.70010 Q4 = 17.1012 Q5 = 38.5442

Silent Run:

Eigenvalues of matrix G: 0.4718 0.4858 0.7164 0.9475 0.9791 Regularization parameter epsilon: 1e-06 Relative error in concentration data: 0.0000 (0.00%) Real Rates Q = [10, 35, 3, 15, 20];	Qepsilon Recovered Rates (via SVD): Q1 = 11.8433 Q2 = 39.8612 Q3 = 2.50267 Q4 = 17.5918 Q5 = 37.1875

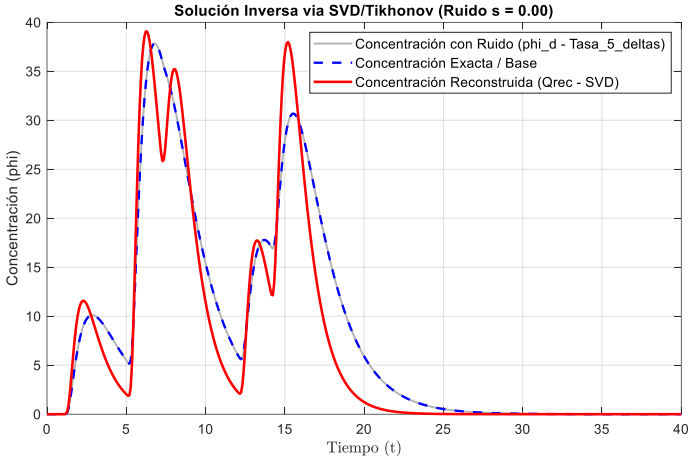


Figure 3: Solution of the Inverse Method via SVD/ Tikhonov

With the largest regulation factor (epsilon) and the weights of the adjoint analytical solution Ci

Eigenvalues of matrix G: 0.9173 1.0708 2.4191 3.9262 4.6890 epsilon regularization parameter : 3.00e-01 Relative error in concentration data (|| delta_phi || / ||phi||): 0.0395 (3.95%) Relative error in recovered rates Q (|| delta_Q || / ||Q||): 0.1289 (12.89%)	Qepsilon Recovered Rates: Q = [10, 35, 3, 15, 20] Q1 = 9.7619 Q2 = 30.2829 Q3 = 5.7939 Q4 = 13.9417 Q5 = 18.8479

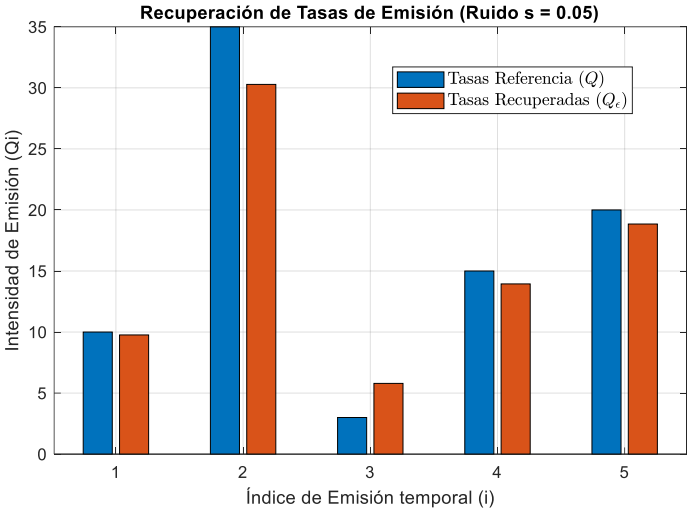


Figure 3: Solution of the Inverse Method via SVD/ Tikhonov

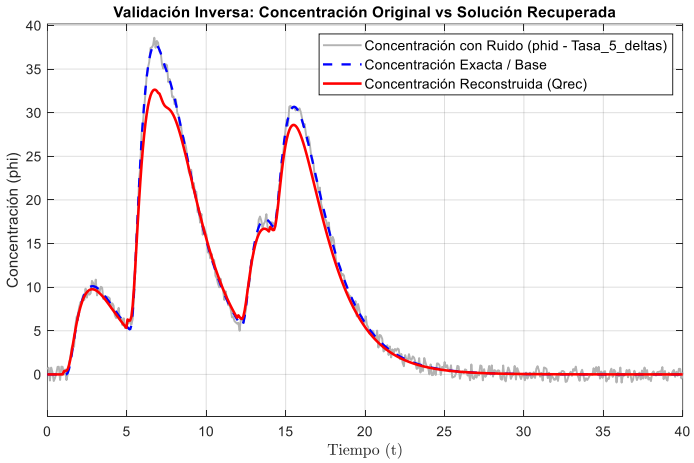


Figure 4: Solution of the Inverse Method via SVD/ Tikhonov

Eigenvalues of matrix G: 0.9173 1.0708 2.4191 3.9262 4.6890 epsilon regularization parameter: 6.00e-01 Relative error in concentration data (|| delta_phi || / ||phi||): 0.0395 (3.95%) Relative error in recovered rates Q (|| delta_Q || / ||Q||): 0.2158 (21.58%)	Qepsilon Recovered Rates: Q = [10, 35, 3, 15, 20] Q1 = 9.4354 Q2 = 26.9472 Q3 = 7.0598 Q4 = 13.3681 Q5 = 17.3663

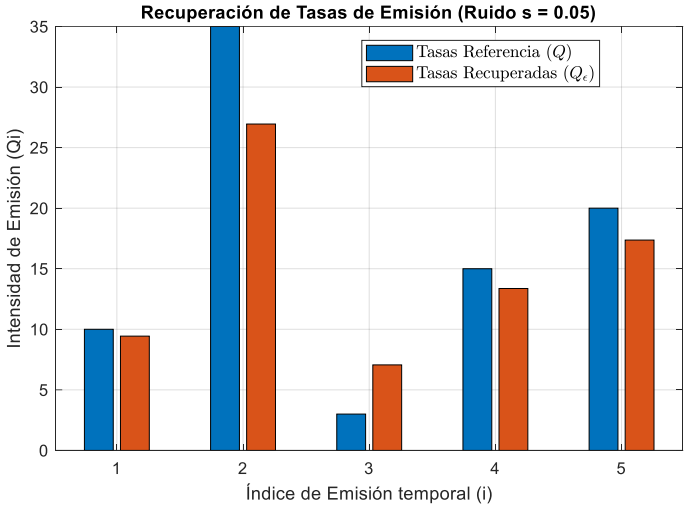


Figure 5: Solution of the Inverse Method via SVD/ Tikhonov

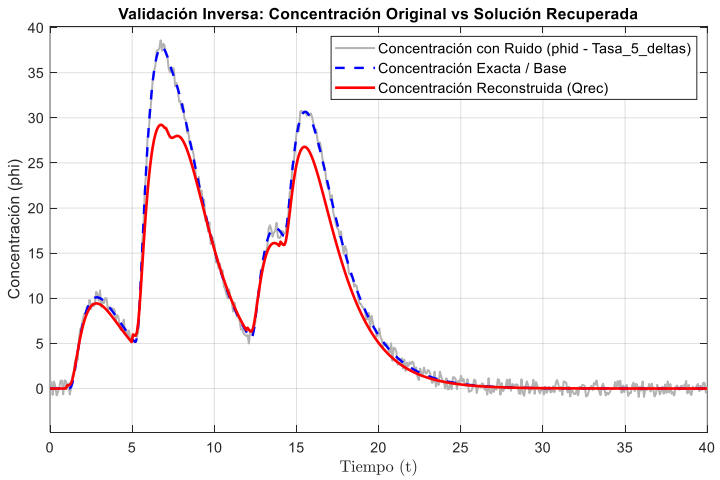
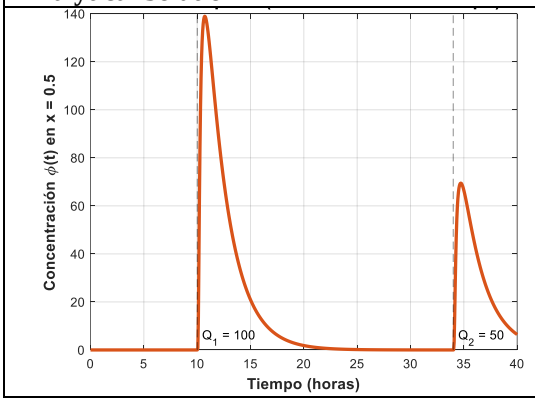
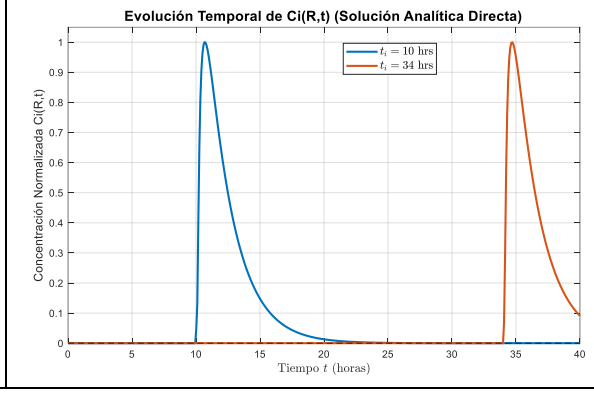


Figure 6: Solution of the Inverse Method via SVD/ Tikhonov

Second Inverse Example: In this example, the homogeneous solution is considered, varying the regulation value epsilon, with rates Q = [100, 50], at tj = [10, 34]

Wind speed	**or**	**0.25**
Diffusion coefficient	**μ**	**0.05**
Transformation rate	**σ**	**0.001**
Domain length	**L**	**1**
Broadcast site	**x_0**	**0.2**
Monitoring site	**R**	**0.5**

Analytical Solution	Adjoint Weights – Adjoint Model
	

Homogeneous Solution

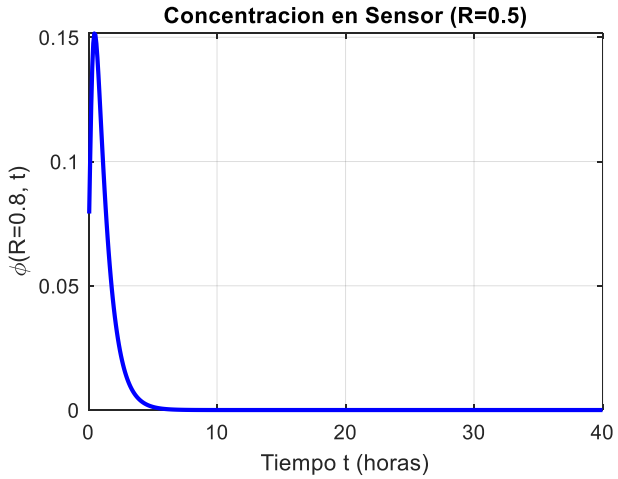


Figure 7: Solution of the Direct Homogeneous Method

========== RESULTS OF THE REVERSE METHOD (SVD) ==========

Eigenvalues of matrix G: 1.7260 1.7360 epsilon regularization parameter: 1e-06 Relative error in concentration data: 0.0000 (0.00%) Relative error in recovered rates Q: 0.3930 (39.30%)	Qepsilon Recovered Rates (via SVD): Q = [100, 50] Q1 = 139.288 Q2 = 69.6678

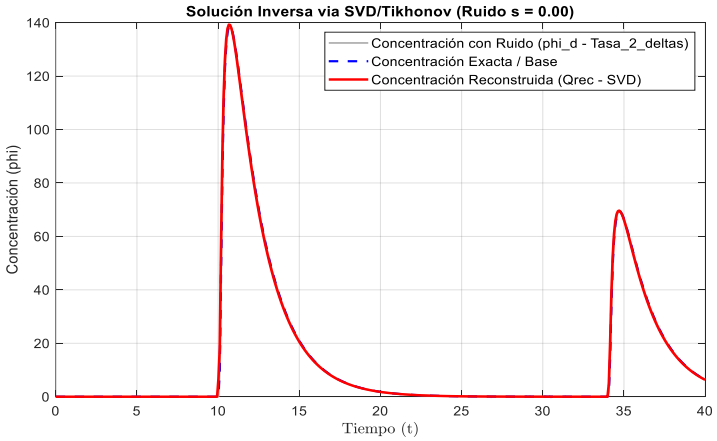


Figure 8: Solution of the Inverse Method via SVD/ Tikhonov

========== RESULTS OF THE INVERSE METHOD (SVD) ==========

Eigenvalues of matrix G: 1.7260 1.7360 Regularization parameter epsilon: 0.00e+00 Relative error in concentration data: 0.0604 (6.04%) Relative error in recovered rates Q: 0.3885 (38.85%)	Qepsilon Recovered Rates (via SVD): Q = [100, 50] Q1 = 138.641 Q2 = 69.8262

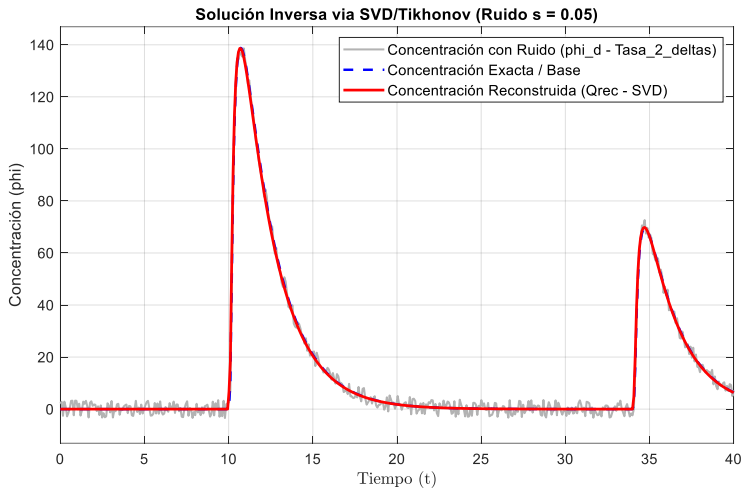


Figure 9: Solution of the Inverse Method via SVD/ Tikhonov

With the larger regulation factor (epsilon) and the solution weights of the adjoint analytical solution Ci

========== ===== RESULTS OF THE REVERSE METHOD (SVD) ==========

Eigenvalues of matrix G: 1.7260 1.7360 epsilon regularization parameter: 6.00e-01 Relative error in concentration data: 0.0604 (6.04%) Relative error in recovered rates Q: 0.0316 (3.16%)	Qepsilon Recovered Rates (via SVD): Q = [100, 50] Q1 = 103.0317 Q2 = 51.8147

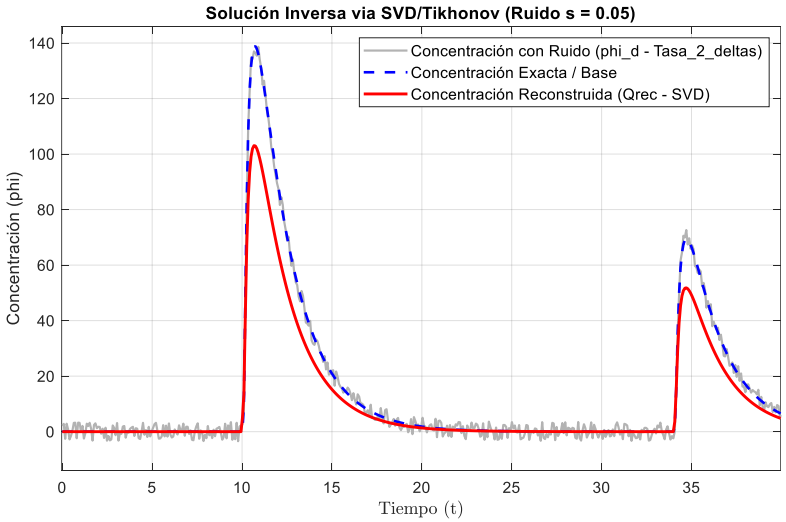


Figure 10: Solution of the Inverse Method via SVD/ Tikhonov

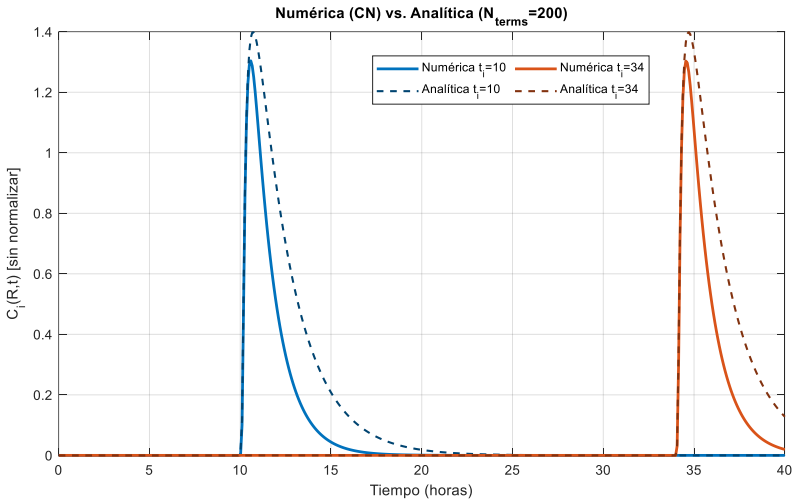


Figure 11: Error validations between analytical and numerical adjoint weights

Conclusions

The method given with the original methodology is already a very robust and well-established method. The method based on this variation is part of how to also address this inversion problem with the classic Tikhonov regulation . It can be seen that by varying the epsilon regulation value, the true rate at which it was released into the atmosphere can be obtained. Direct models change this based on wind speed, diffusion and reaction coefficients, as well as the observation source R with respect to x0. The attached model is the important and crucial part here. As we can see, these weights are what, in a way, dictate the stable or unstable recovery of the delta-type rate(s). From the outset, this type of problem is already poorly formulated, as can be seen. Therefore, the more the error in the simulation is reduced, the better the real rate will be obtained.

I am grateful for the knowledge I have gained from Doctors David Parra Guevara and Yuri N. Skibaof UNAM Mexico City, and I also dedicate this article to the memory of my mother, Ana María Jiménez Castellanos.

References

[1]. Estimation of the intensity of the successive instantaneous emissions released by an air pollution point source International Journal of Applied MathematicsArticleOpenAccess2021 DOI: 10.12732/ ijam.v 34i6.6 Parra-Guevara D. Skiba , Yu. N. Institute of Atmospheric Sciences and Climate Change National Autonomous University of Mexico Circuito Exterior, Ciudad Universitaria CDMX, CP 04510, Mexico.

[2]. G.I. Marchuk, Mathematical Models in Environmental Problems, Elsevier, New York (1986).

[3]. I.G. Enting, Inverse Problems in Atmospheric Constituent Transport, Cambridge University Press, Cambridge (2002).

[4]. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Dover Publications (1923).

[5]. Mathematical models in environmental problems Marchuk G.I.

[6]. Y. N. Skiba and D. Parra-Guevara, Applications of Adjoint Equations to Problems of Dispersion and Control of Pollutants (Nova Science Publishers, 2015)

[7]. Digital Journal — Mathematics, Education and the Internet (http://tecdigital.tec.ac.cr/revistamatematica/). Vol 18, No 2. March − August , 2018 An analytical solution for the diffusion- advection -reaction equation by means of the Fourier series José R. Jiménez Zenteno

[8]. International Journal of Latest Research in Engineering and Technology (IJLRET) ISSN: 2454-5031 www.ijlret.com || Volume 07 - Issue 10 || October 2021 || PP. 39-55 http://www.ijlret.com/Papers/Vol-07-issue-10/4.B2021200.pdf Another Form of Analytical and Numerical Solution of the Partial Differential Equation of Diffusion - Reaction - Dispersion Advection by means of the Green Function - Fourier Series and an Investment Process

[9]. International Journal of Latest Research in Engineering and Technology (IJLRET) ISSN: 2454-5031 www.ijlret.com || Volume 09 - Issue 08 || August 2023 || PP. 16-36 Analysis of Hyperbolic Models a comparison with Parabolic Diffusion Models for Contaminant Dispersion http://www.ijlret.com/Papers/Vol-09-issue-08/2.B2023328.pdf